Chapter 6

The Derivative of exp and Dynkin's Formula

6.1 The Derivative of the Exponential Map

We know that if $[X, Y] = 0$, then $\exp(X + Y) = \exp(X)\exp(Y)$, but this generally false if X and Y do not commute. For X and Y in a small enough open subset, U, containing 0, we know that exp is a diffeomorphism from U to its image, so the function, $\mu\colon U \times U \to U$, given by

$$\mu(X, Y) = \log(\exp(X)\exp(Y))$$

is well-defined and it turns out that, for U small enough, it is analytic. Thus, it is natural to seek a formula for the Taylor expansion of μ near the origin. This problem was investigated by Campbell (1897/98), Baker (1905) and in a more rigorous fashion by Hausdorff (1906). These authors gave recursive identities expressing the Taylor expansion of μ at the origin and the corresponding result is often referred to as the *Campbell-Baker-Hausdorff Formula*. F. Schur (1891) and Poincaré (1899) also investigated the exponential map, in particular formulae for its derivative and the problem of expressing the function μ. However, it was Dynkin who finally gave an explicit formula (see Section 6.3) in 1947.

The proof that μ is analytic in a suitable domain can be proved using a formula for the derivative of the exponential map, a formula that was obtained by F. Schur and Poincaré. Thus, we begin by presenting such a formula.

First, we introduce a convenient notation. If A is any real (or complex) $n \times n$ matrix, the following formula is clear:

$$\int_0^1 e^{tA} dt = \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!}.$$

If A is invertible, then the right-hand side can be written explicitly as

$$\sum_{k=0}^{\infty} \frac{A^k}{(k+1)!} = A^{-1}(e^A - I),$$

and we also write the latter as

$$\frac{e^A - I}{A} = \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!}. \tag{$*$}$$

Even if A is not invertible, we use $(*)$ as the definition of $\frac{e^A - I}{A}$.

We can use the following trick to figure out what $(d_X \exp)(Y)$ is:

$$(d_X \exp)(Y) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \exp(X + \epsilon Y) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} dR_{\exp(X+\epsilon Y)}(1),$$

since by Proposition 5.2, the map, $s \mapsto R_{\exp s(X+\epsilon Y)}$ is the flow of the left-invariant vector field $(X + \epsilon Y)^L$ on G. Now, $(X + \epsilon Y)^L$ is an ϵ-dependent vector field which depends on ϵ in a C^1 fashion. From the theory of ODE's, if $p \mapsto v_\epsilon(p)$ is a smooth vector field depending in a C^1 fashion on a real parameter ϵ and if Φ_t^ϵ denotes its flow (after time), then the map $\epsilon \mapsto \Phi_t^\epsilon$ is differentiable and we have

$$\frac{\partial \Phi_t^\epsilon}{\partial \epsilon}(x) = \int_0^t d_{\Phi_t^\epsilon(x)}(\Phi_{t-s}^\epsilon) \frac{\partial v_\epsilon}{\partial \epsilon}(\Phi_s^\epsilon(x)) ds.$$

See Duistermaat and Kolk [53], Appendix B, Formula (B.10). Using this, the following is proved in Duistermaat and Kolk [53] (Chapter 1, Section 1.5):

Proposition 6.1 *Given any Lie group, G, for any $X \in \mathfrak{g}$, the linear map, $d\exp_X \colon \mathfrak{g} \to T_{\exp(X)}G$, is given by*

$$\begin{aligned} d\exp_X &= (dR_{\exp(X)})_1 \circ \int_0^1 e^{s\,\mathrm{ad}\,X} ds = (dR_{\exp(X)})_1 \circ \frac{e^{\mathrm{ad}\,X} - I}{\mathrm{ad}\,X} \\ &= (dL_{\exp(X)})_1 \circ \int_0^1 e^{-s\,\mathrm{ad}\,X} ds = (dL_{\exp(X)})_1 \circ \frac{I - e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X}. \end{aligned}$$

Remark: If G is a matrix group of $n \times n$ matrices, we see immediately that the derivative of left multiplication ($X \mapsto L_A X = AX$) is given by

$$(dL_A)_X Y = AY,$$

for all $n \times n$ matrices, X, Y. Consequently, for a matrix group, we get

$$d\exp_X = e^X \left(\frac{I - e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X} \right).$$

Now, if A is a real matrix, it is clear that the (complex) eigenvalues of $\int_0^1 e^{sA} ds$ are of the form

$$\frac{e^\lambda - 1}{\lambda} \quad (= 1 \quad \text{if } \lambda = 0),$$

where λ ranges over the (complex) eigenvalues of A. Consequently, we get

Proposition 6.2 *The singular points of the exponential map,* $\exp\colon \mathfrak{g} \to G$*, that is, the set of* $X \in \mathfrak{g}$ *such that* $d\exp_X$ *is singular (not invertible) are the* $X \in \mathfrak{g}$ *such that the linear map,* $\mathrm{ad}\, X\colon \mathfrak{g} \to \mathfrak{g}$*, has an eigenvalue of the form* $k2\pi i$*, with* $k \in \mathbb{Z}$ *and* $k \neq 0$*.*

Another way to describe the *singular locus*, Σ, of the exponential map is to say that it is the disjoint union

$$\Sigma = \bigcup_{k\in\mathbb{Z}-\{0\}} k\Sigma_1,$$

where Σ_1 is the algebraic variety in $\mathfrak{g}$ given by

$$\Sigma_1 = \{X \in \mathfrak{g} \mid \det(\mathrm{ad}\, X - 2\pi i\, I) = 0\}.$$

For example, for $\mathrm{SL}(2, \mathbb{R})$,

$$\Sigma_1 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2) \mid a^2 + bc = -\pi^2 \right\},$$

a two-sheeted hyperboloid mapped to $-I$ by exp.

Let $\mathfrak{g}_e = \mathfrak{g} - \Sigma$ be the set of $X \in \mathfrak{g}$ such that $\frac{e^{\mathrm{ad}\, X} - I}{\mathrm{ad}\, X}$ is invertible. This is an open subset of $\mathfrak{g}$ containing 0.

6.2 The Product in Logarithmic Coordinates

Since the map,

$$X \mapsto \frac{e^{\mathrm{ad}\, X} - I}{\mathrm{ad}\, X}$$

is invertible for all $X \in \mathfrak{g}_e = \mathfrak{g} - \Sigma$, in view of the chain rule, the inverse of the above map,

$$X \mapsto \frac{\mathrm{ad}\, X}{e^{\mathrm{ad}\, X} - I},$$

is an analytic function from $\mathfrak{g}_e$ to $\mathfrak{gl}(\mathfrak{g}, \mathfrak{g})$. Let $\mathfrak{g}_e^2$ be the subset of $\mathfrak{g} \times \mathfrak{g}_e$ consisting of all (X, Y) such that the solution, $t \mapsto Z(t)$, of the differential equation

$$\frac{dZ(t)}{dt} = \frac{\mathrm{ad}\, Z(t)}{e^{\mathrm{ad}\, Z(t)} - I}(X)$$

with initial condition $Z(0) = Y (\in \mathfrak{g}_e)$, is defined for all $t \in [0, 1]$. Set

$$\mu(X, Y) = Z(1), \quad (X, Y) \in \mathfrak{g}_e^2.$$

The following theorem is proved in Duistermaat and Kolk [53] (Chapter 1, Section 1.6, Theorem 1.6.1):

Theorem 6.3 *Given any Lie group G with Lie algebra, $\mathfrak{g}$, the set $\mathfrak{g}_e^2$ is an open subset of $\mathfrak{g}\times\mathfrak{g}$ containing $(0,0)$ and the map, $\mu\colon \mathfrak{g}_e^2 \to \mathfrak{g}$, is real-analytic. Furthermore, we have*

$$\exp(X)\exp(Y) = \exp(\mu(X,Y)), \qquad (X,Y)\in \mathfrak{g}_e^2,$$

where $\exp\colon \mathfrak{g}\to G$. If $\mathfrak{g}$ is a complex Lie algebra, then μ is complex-analytic.

We may think of μ as the product in logarithmic coordinates. It is explained in Duistermaat and Kolk [53] (Chapter 1, Section 1.6) how Theorem 6.3 implies that a Lie group can be provided with the structure of a real-analytic Lie group. Rather than going into this, we will state a remarkable formula due to Dynkin expressing the Taylor expansion of μ at the origin.

6.3 Dynkin's Formula

As we said in Section 6.3, the problem of finding the Taylor expansion of μ near the origin was investigated by Campbell (1897/98), Baker (1905) and Hausdorff (1906). However, it was Dynkin who finally gave an explicit formula in 1947. There are actually slightly different versions of Dynkin's formula. One version is given (and proved convergent) in Duistermaat and Kolk [53] (Chapter 1, Section 1.7). Another slightly more explicit version (because it gives a formula for the homogeneous components of $\mu(X,Y)$) is given (and proved convergent) in Bourbaki [22] (Chapter II, §6, Section 4) and Serre [136] (Part I, Chapter IV, Section 8). We present the version in Bourbaki and Serre without proof. The proof uses formal power series and free Lie algebras.

Given $X, Y \in \mathfrak{g}_e^2$, we can write

$$\mu(X,Y) = \sum_{n=1}^{\infty} z_n(X,Y),$$

where $z_n(X,Y)$ is a homogeneous polynomial of degree n in the non-commuting variables X, Y.

Theorem 6.4 *(Dynkin's Formula) If we write $\mu(X,Y)=\sum_{n=1}^{\infty} z_n(X,Y)$, then we have*

$$z_n(X,Y) = \frac{1}{n}\sum_{p+q=n}(z'_{p,q}(X,Y) + z''_{p,q}(X,Y)),$$

with

$$z'_{p,q}(X,Y) = \sum_{\substack{p_1+\cdots+p_m=p\\ q_1+\cdots+q_{m-1}=q-1\\ p_i+q_i\geq 1,\, p_m\geq 1,\, m\geq 1}} \frac{(-1)^{m+1}}{m}\left(\left(\prod_{i=1}^{m-1}\frac{(\mathrm{ad}\, X)^{p_i}}{p_i!}\frac{(\mathrm{ad}\, Y)^{q_i}}{q_i!}\right)\frac{(\mathrm{ad}\, X)^{p_m}}{p_m!}\right)(Y)$$

and

$$z''_{p,q}(X,Y) = \sum_{\substack{p_1+\cdots+p_{m-1}=p-1\\ q_1+\cdots+q_{m-1}=q\\ p_i+q_i\geq 1,\, m\geq 1}} \frac{(-1)^{m+1}}{m}\left(\prod_{i=1}^{m-1}\frac{(\mathrm{ad}\,X)^{p_i}}{p_i!}\frac{(\mathrm{ad}\,Y)^{q_i}}{q_i!}\right)(X).$$

As a concrete illustration of Dynkin's formula, after some labor, the following Taylor expansion up to order 4 is obtained:

$$\begin{aligned}\mu(X,Y) \;&=\; X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]-\frac{1}{24}[X,[Y,[X,Y]]]\\ &\quad + \text{higher order terms.}\end{aligned}$$

Observe that due the lack of associativity of the Lie bracket quite different looking expressions can be obtained using the Jacobi identity. For example,

$$-[X,[Y,[X,Y]]] = [Y,[X,[Y,X]]].$$

There is also an integral version of the Campbell-Baker-Hausdorff formula, see Hall [70] (Chapter 3).

Chapter 7

Bundles, Riemannian Manifolds and Homogeneous Spaces, II

7.1 Fibre Bundles

We saw in Section 2.2 that a transitive action, $\cdot\colon G \times X \to X$, of a group, G, on a set, X, yields a description of X as a quotient G/G_x, where G_x is the stabilizer of any element, $x \in X$. In Theorem 2.26, we saw that if X is a "well-behaved" topological space, G is a "well-behaved" topological group and the action is continuous, then G/G_x is homeomorphic to X. In particular the conditions of Theorem 2.26 are satisfied if G is a Lie group and X is a manifold. Intuitively, the above theorem says that G can be viewed as a family of "fibres", G_x, all isomorphic to G, these fibres being parametrized by the "base space", X, and varying smoothly when x moves in X. We have an example of what is called a fibre bundle, in fact, a principal fibre bundle. Now that we know about manifolds and Lie groups, we can be more precise about this situation.

Although we will not make extensive use of it, we begin by reviewing the definition of a fibre bundle because we believe that it clarifies the notions of vector bundles and principal fibre bundles, the concepts that are our primary concern. The following definition is not the most general but it is sufficient for our needs:

Definition 7.1 A *fibre bundle with (typical) fibre, F, and structure group, G,* is a tuple, $\xi = (E, \pi, B, F, G)$, where E, B, F are smooth manifolds, $\pi\colon E \to B$ is a smooth surjective map, G is a Lie group of diffeomorphisms of F and there is some open cover, $\mathcal{U} = (U_\alpha)_{\alpha\in I}$, of B and a family, $\varphi = (\varphi_\alpha)_{\alpha\in I}$, of diffeomorphisms,

$$\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F.$$

The space, B, is called the *base space*, E is called the *total space*, F is called the *(typical) fibre*, and each φ_α is called a *(local) trivialization*. The pair, $(U_\alpha, \varphi_\alpha)$, is called a *bundle chart* and the family, $\{(U_\alpha, \varphi_\alpha)\}$, is a *trivializing cover*. For each $b \in B$, the space, $\pi^{-1}(b)$, is called the *fibre above* b; it is also denoted by E_b, and $\pi^{-1}(U_\alpha)$ is also denoted by $E \upharpoonright U_\alpha$. Furthermore, the following properties hold:

(a) The diagram

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "p_1"] \\ & U_\alpha & \end{tikzcd}$$

commutes for all $\alpha \in I$, where $p_1 \colon U_\alpha \times F \to U_\alpha$ is the first projection. Equivalently, for all $(b, y) \in U_\alpha \times F$,

$$\pi \circ \varphi_\alpha^{-1}(b, y) = b.$$

For every $(U_\alpha, \varphi_\alpha)$ and every $b \in U_\alpha$, we have the diffeomorphism,

$$(p_2 \circ \varphi_\alpha) \upharpoonright E_b \colon E_b \to F,$$

where $p_2 \colon U_\alpha \times F \to F$ is the second projection, which we denote by $\varphi_{\alpha,b}$. (So, we have the diffeomorphism, $\varphi_{\alpha,b} \colon \pi^{-1}(b)\ (= E_b) \to F$.) Furthermore, for all U_α, U_β in $\mathcal{U}$ such that $U_\alpha \cap U_\beta \neq \emptyset$, for every $b \in U_\alpha \cap U_\beta$, there is a relationship between $\varphi_{\alpha,b}$ and $\varphi_{\beta,b}$ which gives the twisting of the bundle:

(b) The diffeomorphism,

$$\varphi_{\alpha,b} \circ \varphi_{\beta,b}^{-1} \colon F \to F,$$

is an element of the group G.

(c) The map, $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$, defined by

$$g_{\alpha\beta}(b) = \varphi_{\alpha,b} \circ \varphi_{\beta,b}^{-1}$$

is smooth. The maps $g_{\alpha\beta}$ are called the *transition maps* of the fibre bundle.

A fibre bundle, $\xi = (E, \pi, B, F, G)$, is also referred to, somewhat loosely, as a *fibre bundle over* B or a G*-bundle* and it is customary to use the notation

$$F \longrightarrow E \longrightarrow B,$$

or

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

even though it is imprecise (the group G is missing!) and it clashes with the notation for short exact sequences. Observe that the bundle charts, $(U_\alpha, \varphi_\alpha)$, are similar to the charts of a manifold.

Actually, Definition 7.1 is too restrictive because it does not allow for the addition of compatible bundle charts, for example, when considering a refinement of the cover, $\mathcal{U}$. This problem can easily be fixed using a notion of equivalence of trivializing covers analogous to the equivalence of atlases for manifolds (see Remark (2) below). Also Observe that (b) and (c) imply that the isomorphism, $\varphi_\alpha \circ \varphi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$, is related to the smooth map, $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$, by the identity

$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, x) = (b, g_{\alpha\beta}(b)(x)),$$

for all $b \in U_\alpha \cap U_\beta$ and all $x \in F$.

Note that the isomorphism, $\varphi_\alpha \circ \varphi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$, describes how the fibres viewed over U_β are viewed over U_α. Thus, it might have been better to denote $g_{\alpha,\beta}$ by g_β^α, so that

$$g_\alpha^\beta = \varphi_{\beta,b} \circ \varphi_{\alpha,b}^{-1},$$

where the subscript, α, indicates the source and the superscript, β, indicates the target.

Intuitively, a fibre bundle over B is a family, $E = (E_b)_{b \in B}$, of spaces, E_b, (fibres) indexed by B and varying smoothly as b moves in B, such that every E_b is diffeomorphic to F. The bundle, $E = B \times F$, where π is the first projection, is called the *trivial bundle* (over B). The trivial bundle, $B \times F$, is often denoted ϵ^F. The local triviality condition (a) says that *locally*, that is, over every subset, U_α, from some open cover of the base space, B, the bundle $\xi \restriction U_\alpha$ is trivial. Note that if G is the trivial one-element group, then the fibre bundle is trivial. In fact, the purpose of the group G is to specify the "twisting" of the bundle, that is, how the fibre, E_b, gets twisted as b moves in the base space, B.

A Möbius strip is an example of a nontrivial fibre bundle where the base space, B, is the circle S^1 and the fibre space, F, is the closed interval $[-1, 1]$ and the structural group is $G = \{1, -1\}$, where -1 is the reflection of the interval $[-1, 1]$ about its midpoint, 0. The total space, E, is the strip obtained by rotating the line segment $[-1, 1]$ around the circle, keeping its midpoint in contact with the circle, and gradually twisting the line segment so that after a full revolution, the segment has been tilted by π. The reader should work out the transition functions for an open cover consisting of two open intervals on the circle.

A Klein bottle is also a fibre bundle for which both the base space and the fibre are the circle, S^1. Again, the reader should work out the details for this example.

Other examples of fibre bundles are:

(1) $\mathbf{SO}(n+1)$, an $\mathbf{SO}(n)$-bundle over the sphere S^n with fibre $\mathbf{SO}(n)$. (for $n \geq 0$).

(2) $\mathbf{SU}(n+1)$, an $\mathbf{SU}(n)$-bundle over the sphere S^{2n+1} with fibre $\mathbf{SU}(n)$ (for $n \geq 0$).

(3) $\mathbf{SL}(2, \mathbb{R})$, an $\mathbf{SO}(2)$-bundle over the upper-half space H, with fibre $\mathbf{SO}(2)$.

(4) $\mathbf{GL}(n, \mathbb{R})$, an $\mathbf{O}(n)$-bundle over the space, $\mathbf{SPD}(n)$, of symmetric, positive definite matrices, with fibre $\mathbf{O}(n)$.

(5) $\mathbf{GL}^+(n,\mathbb{R})$, an $\mathbf{SO}(n)$-bundle over the space, $\mathbf{SPD}(n)$, of symmetric, positive definite matrices, with fibre $\mathbf{SO}(n)$.

(6) $\mathbf{SO}(n+1)$, an $\mathbf{O}(n)$-bundle over the real projective space $\mathbb{RP}^n$ with fibre $\mathbf{O}(n)$ (for $n \geq 0$).

(7) $\mathbf{SU}(n+1)$, an $\mathbf{U}(n)$-bundle over the complex projective space $\mathbb{CP}^n$ with fibre $\mathbf{U}(n)$ (for $n \geq 0$).

(8) $\mathbf{O}(n)$, an $\mathbf{O}(k) \times \mathbf{O}(n-k)$-bundle over the Grassmannian, $G(k,n)$ with fibre $\mathbf{O}(k) \times \mathbf{O}(n-k)$.

(9) $\mathbf{SO}(n)$ an $S(\mathbf{O}(k) \times \mathbf{O}(n-k))$-bundle over the Grassmannian, $G(k,n)$ with fibre $S(\mathbf{O}(k) \times \mathbf{O}(n-k))$.

(10) From Section 2.5, we see that the Lorentz group, $\mathbf{SO}_0(n,1)$, is an $\mathbf{SO}(n)$-bundle over the space, $\mathcal{H}_n^+(1)$, consisting of one sheet of the hyperbolic paraboloid, $\mathcal{H}_n(1)$, with fibre $\mathbf{SO}(n)$.

Observe that in all the examples above, $F = G$, that is, the typical fibre is identical to the group G. Special bundles of this kind are called *principal fibre bundles*.

Remarks:

(1) The above definition is slightly different (but equivalent) to the definition given in Bott and Tu [19], page 47-48. Definition 7.1 is closer to the one given in Hirzebruch [77]. Bott and Tu and Hirzebruch assume that G acts effectively on the left on the fibre, F. This means that there is a smooth action, $\cdot\colon G \times F \to F$, and recall that G acts effectively on F iff for every $g \in G$,

$$\text{if} \quad g \cdot x = x \quad \text{for all } x \in F, \quad \text{then} \quad g = 1.$$

Every $g \in G$ induces a diffeomorphism, $\varphi_g\colon F \to F$, defined by

$$\varphi_g(x) = g \cdot x,$$

for all $x \in F$. The fact that G acts effectively on F means that the map, $g \mapsto \varphi_g$, is injective. This justifies viewing G as a group of diffeomorphisms of F, and from now on, we will denote $\varphi_g(x)$ by $g(x)$.

(2) We observed that Definition 7.1 is too restrictive because it does not allow for the addition of compatible bundle charts. We can fix this problem as follows: Given a trivializing cover, $\{(U_\alpha, \varphi_\alpha)\}$, for any open, U, of B and any diffeomorphism,

$$\varphi\colon \pi^{-1}(U) \to U \times F,$$

we say that (U, φ) is *compatible with the trivializing cover*, $\{(U_\alpha, \varphi_\alpha)\}$, iff whenever $U \cap U_\alpha \neq \emptyset$, there is some smooth map, $g_\alpha \colon U \cap U_\alpha \to G$, so that

$$\varphi \circ \varphi_\alpha^{-1}(b, x) = (b, g_\alpha(b)(x)),$$

for all $b \in U \cap U_\alpha$ and all $x \in F$. Two trivializing covers are *equivalent* iff every bundle chart of one cover is compatible with the other cover. This is equivalent to saying that the union of two trivializing covers is a trivializing cover. Then, we can define a fibre bundle as a tuple, $(E, \pi, B, F, G, \{(U_\alpha, \varphi_\alpha)\})$, where $\{(U_\alpha, \varphi_\alpha)\}$ is an equivalence class of trivializing covers. As for manifolds, given a trivializing cover, $\{(U_\alpha, \varphi_\alpha)\}$, the set of all bundle charts compatible with $\{(U_\alpha, \varphi_\alpha)\}$ is a maximal trivializing cover equivalent to $\{(U_\alpha, \varphi_\alpha)\}$.

A special case of the above occurs when we have a trivializing cover, $\{(U_\alpha, \varphi_\alpha)\}$, with $\mathcal{U} = \{U_\alpha\}$ an open cover of B and another open cover, $\mathcal{V} = (V_\beta)_{\beta \in J}$, of B which is a refinement of $\mathcal{U}$. This means that there is a map, $\tau \colon J \to I$, such that $V_\beta \subseteq U_{\tau(\beta)}$ for all $\beta \in J$. Then, for every $V_\beta \in \mathcal{V}$, since $V_\beta \subseteq U_{\tau(\beta)}$, the restriction of $\varphi_{\tau(\beta)}$ to V_β is a trivialization

$$\varphi'_\beta \colon \pi^{-1}(V_\beta) \to V_\beta \times F$$

and conditions (b) and (c) are still satisfied, so $(V_\beta, \varphi'_\beta)$ is compatible with $\{(U_\alpha, \varphi_\alpha)\}$.

(3) (For readers familiar with sheaves) Hirzebruch defines the sheaf, G_∞, where $\Gamma(U, G_\infty)$ is the group of smooth functions, $g \colon U \to G$, where U is some open subset of B and G is a Lie group acting effectively (on the left) on the fibre F. The group operation on $\Gamma(U, G_\infty)$ is induced by multiplication in G, that is, given two (smooth) functions, $g \colon U \to G$ and $h \colon U \to G$,

$$gh(b) = g(b)h(b),$$

for all $b \in U$.

Beware that gh is **not** function composition, unless G itself is a group of functions, which is the case for vector bundles.

Our conditions (b) and (c) are then replaced by the following equivalent condition: For all U_α, U_β in $\mathcal{U}$ such that $U_\alpha \cap U_\beta \neq \emptyset$, there is some $g_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, G_\infty)$ such that

$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, x) = (b, g_{\alpha\beta}(b)(x)),$$

for all $b \in U_\alpha \cap U_\beta$ and all $x \in F$.

(4) The family of transition functions $(g_{\alpha\beta})$ satisfies the *cocycle condition*,

$$g_{\alpha\beta}(b) g_{\beta\gamma}(b) = g_{\alpha\gamma}(b),$$

for all α, β, γ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ and all $b \in U_\alpha \cap U_\beta \cap U_\gamma$. Setting $\alpha = \beta = \gamma$, we get

$$g_{\alpha\alpha} = \mathrm{id},$$

and setting $\gamma = \alpha$, we get

$$g_{\beta\alpha} = g_{\alpha\beta}^{-1}.$$

Again, beware that this means that $g_{\beta\alpha}(b) = g_{\alpha\beta}^{-1}(b)$, where $g_{\alpha\beta}^{-1}(b)$ is the inverse of $g_{\beta\alpha}(b)$ in G. In general, $g_{\alpha\beta}^{-1}$ is **not** the functional inverse of $g_{\beta\alpha}$.

The classic source on fibre bundles is Steenrod [141]. The most comprehensive treatment of fibre bundles and vector bundles is probably given in Husemoller [82]. However, we can hardly recommend this book. We find the presentation overly formal and intuitions are absent. A more extensive list of references is given at the end of Section 7.5.

Remark: (The following paragraph is intended for readers familiar with Čech cohomology.) The cocycle condition makes it possible to view a fibre bundle over B as a member of a certain *(Čech) cohomology set*, $\check{H}^1(B, \mathcal{G})$, where $\mathcal{G}$ denotes a certain sheaf of functions from the manifold B into the Lie group G, as explained in Hirzebruch [77], Section 3.2. However, this requires defining a noncommutative version of Čech cohomology (at least, for $\check{H}^1$), and clarifying when two open covers and two trivializations define the same fibre bundle over B, or equivalently, defining when two fibre bundles over B are equivalent. If the bundles under considerations are line bundles (see Definition 7.6), then $\check{H}^1(B, \mathcal{G})$ is actually a group. In this case, $G = \mathrm{GL}(1, \mathbb{R}) \cong \mathbb{R}^*$ in the real case and $G = \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$ in the complex case (where $\mathbb{R}^* = \mathbb{R} - \{0\}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$), and the sheaf $\mathcal{G}$ is the sheaf of smooth (real-valued or complex-valued) functions vanishing nowhere. The group, $\check{H}^1(B, \mathcal{G})$, plays an important role, especially when the bundle is a holomorphic line bundle over a complex manifold. In the latter case, it is called the *Picard group* of B.

The notion of a map between fibre bundles is more subtle than one might think because of the structure group, G. Let us begin with the simpler case where $G = \mathrm{Diff}(F)$, the group of all smooth diffeomorphisms of F.

Definition 7.2 If $\xi_1 = (E_1, \pi_1, B_1, F, \mathrm{Diff}(F))$ and $\xi_2 = (E_2, \pi_2, B_2, F, \mathrm{Diff}(F))$ are two fibre bundles with the same typical fibre, F, and the same structure group, $G = \mathrm{Diff}(F)$, a *bundle map (or bundle morphism)*, $f \colon \xi_1 \to \xi_2$, is a pair, $f = (f_E, f_B)$, of smooth maps, $f_E \colon E_1 \to E_2$ and $f_B \colon B_1 \to B_2$, such that

(a) The following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f_B"'] & B_2 \end{tikzcd}$$

(b) For every $b \in B_1$, the map of fibres,

$$f_E \restriction \pi_1^{-1}(b) \colon \pi_1^{-1}(b) \to \pi_2^{-1}(f_B(b)),$$

is a diffeomorphism (*preservation of the fibre*).

A bundle map, $f\colon \xi_1 \to \xi_2$, is an *isomorphism* if there is some bundle map, $g\colon \xi_2 \to \xi_1$, called the *inverse of* f such that

$$g_E \circ f_E = \mathrm{id} \quad \text{and} \quad f_E \circ g_E = \mathrm{id}.$$

The bundles ξ_1 and ξ_2 are called *isomorphic*. Given two fibre bundles, $\xi_1 = (E_1, \pi_1, B, F, G)$ and $\xi_2 = (E_2, \pi_2, B, F, G)$, over the same base space, B, a *bundle map (or bundle morphism)*, $f\colon \xi_1 \to \xi_2$, is a pair, $f = (f_E, f_B)$, where $f_B = \mathrm{id}$ (the identity map). Such a bundle map is an *isomorphism* if it has an inverse as defined above. In this case, we say that the bundles ξ_1 and ξ_2 over B are *isomorphic*.

Observe that the commutativity of the diagram in Definition 7.2 implies that f_B is actually determined by f_E. Also, when f is an isomorphism, the surjectivity of π_1 and π_2 implies that

$$g_B \circ f_B = \mathrm{id} \quad \text{and} \quad f_B \circ g_B = \mathrm{id}.$$

Thus, when $f = (f_E, f_B)$ is an isomorphism, both f_E and f_B are diffeomorphisms.

Remark: Some authors do not require the "preservation" of fibres. However, it is automatic for bundle isomorphisms.

When we have a bundle map, $f\colon \xi_1 \to \xi_2$, as above, for every $b \in B$, for any trivializations $\varphi_\alpha\colon \pi_1^{-1}(U_\alpha) \to U_\alpha \times F$ of ξ_1 and $\varphi'_\beta\colon \pi_2^{-1}(V_\beta) \to V_\beta \times F$ of ξ_2, with $b \in U_\alpha$ and $f_B(b) \in V_\beta$, we have the map,

$$\varphi'_\beta \circ f_E \circ \varphi_\alpha^{-1} \colon (U_\alpha \cap f_B^{-1}(V_\beta)) \times F \to V_\beta \times F.$$

Consequently, as φ_α and φ'_α are diffeomorphisms and as f is a diffeomorphism on fibres, we have a map, $\rho_{\alpha,\beta}\colon U_\alpha \cap f_B^{-1}(V_\beta) \to \mathrm{Diff}(F)$, such that

$$\varphi'_\beta \circ f_E \circ \varphi_\alpha^{-1}(b, x) = (f_B(b), \rho_{\alpha,\beta}(b)(x)),$$

for all $b \in U_\alpha \cap f_B^{-1}(V_\beta)$ and all $x \in F$. Unfortunately, in general, there is no garantee that $\rho_{\alpha,\beta}(b) \in G$ or that it be smooth. However, this will be the case when ξ is a vector bundle or a principal bundle.

Since we may always pick U_α and V_β so that $f_B(U_\alpha) \subseteq V_\beta$, we may also write ρ_α instead of $\rho_{\alpha,\beta}$, with $\rho_\alpha\colon U_\alpha \to G$. Then, observe that locally, f_E is given as the composition

$$\begin{array}{ccccccc}
\pi_1^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times F & \xrightarrow{\widetilde{f}_\alpha} & V_\beta \times F & \xrightarrow{{\varphi'_\beta}^{-1}} & \pi_2^{-1}(V_\beta) \\
z & \longrightarrow & (b, x) & \longrightarrow & (f_B(b), \rho_\alpha(b)(x)) & \longrightarrow & {\varphi'_\beta}^{-1}(f_B(b), \rho_\alpha(b)(x)),
\end{array}$$

with $\widetilde{f}_\alpha(b, x) = (f_B(b), \rho_\alpha(b)(x))$, that is,

$$f_E(z) = {\varphi'_\beta}^{-1}(f_B(b), \rho_\alpha(b)(x)), \qquad \text{with } z \in \pi_1^{-1}(U_\alpha) \text{ and } (b, x) = \varphi_\alpha(z).$$

Conversely, if (f_E, f_B) is a pair of smooth maps satisfying the commutative diagram of Definition 7.2 and the above conditions hold locally, then as φ_α, ${\varphi'_\beta}^{-1}$ and $\rho_\alpha(b)$ are diffeomorphisms on fibres, we see that f_E is a diffeomorphism on fibres.

In the general case where the structure group, G, is not the whole group of diffeomorphisms, $\mathrm{Diff}(F)$, following Hirzebruch [77], we use the local conditions above to define the "right notion" of bundle map, namely Definition 7.3. Another advantage of this definition is that two bundles (with the same fibre, structure group, and base) are isomorphic iff they are equivalent (see Proposition 7.1 and Proposition 7.2).

Definition 7.3 Given two fibre bundles, $\xi_1 = (E_1, \pi_1, B_1, F, G)$ and $\xi_2 = (E_2, \pi_2, B_2, F, G)$, a *bundle map*, $f\colon \xi_1 \to \xi_2$, is a pair, $f = (f_E, f_B)$, of smooth maps, $f_E\colon E_1 \to E_2$ and $f_B\colon B_1 \to B_2$, such that

(a) The diagram

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f_B"'] & B_2 \end{tikzcd}$$

commutes.

(b) There is an open cover, $\mathcal{U} = (U_\alpha)_{\alpha\in I}$, for B_1, an open cover, $\mathcal{V} = (V_\beta)_{\beta\in J}$, for B_2, a family, $\varphi = (\varphi_\alpha)_{\alpha\in I}$, of trivializations, $\varphi_\alpha\colon \pi_1^{-1}(U_\alpha) \to U_\alpha \times F$, for ξ_1, a family, $\varphi' = (\varphi'_\beta)_{\beta\in J}$, of trivializations, $\varphi'_\beta\colon \pi_2^{-1}(V_\beta) \to V_\beta \times F$, for ξ_2, such that for every $b \in B$, there are some trivializations, $\varphi_\alpha\colon \pi_1^{-1}(U_\alpha) \to U_\alpha \times F$ and $\varphi'_\beta\colon \pi_2^{-1}(V_\beta) \to V_\beta \times F$, with $f_B(U_\alpha) \subseteq V_\beta$, $b \in U_\alpha$ and some smooth map,

$$\rho_\alpha\colon U_\alpha \to G,$$

such that $\varphi'_\beta \circ f_E \circ \varphi_\alpha^{-1}\colon U_\alpha \times F \to V_\alpha \times F$ is given by

$$\varphi'_\beta \circ f_E \circ \varphi_\alpha^{-1}(b, x) = (f_B(b), \rho_\alpha(b)(x)),$$

for all $b \in U_\alpha$ and all $x \in F$.

A bundle map is an *isomorphism* if it has an inverse as in Definition 7.2. If the bundles ξ_1 and ξ_2 are over the same base, B, then we also require $f_B = \mathrm{id}$.

As we remarked in the discussion before Definition 7.3, condition (b) insures that the maps of fibres,

$$f_E \restriction \pi_1^{-1}(b)\colon \pi_1^{-1}(b) \to \pi_2^{-1}(f_B(b)),$$

are diffeomorphisms. In the special case where ξ_1 and ξ_2 have the same base, $B_1 = B_2 = B$, we require $f_B = \mathrm{id}$ and we can use the same cover (*i.e.*, $\mathcal{U} = \mathcal{V}$) in which case condition (b) becomes: There is some smooth map, $\rho_\alpha \colon U_\alpha \to G$, such that

$$\varphi'_\alpha \circ f \circ {\varphi_\alpha}^{-1}(b, x) = (b, \rho_\alpha(b)(x)),$$

for all $b \in U_\alpha$ and all $x \in F$.

We say that a bundle, ξ, with base B and structure group G is trivial iff ξ is isomorphic to the product bundle, $B \times F$, according to the notion of isomorphism of Definition 7.3.

We can also define the notion of equivalence for fibre bundles over the same base space, B (see Hirzebruch [77], Section 3.2, Chern [33], Section 5, and Husemoller [82], Chapter 5). We will see shortly that two bundles over the same base are equivalent iff they are isomorphic.

Definition 7.4 Given two fibre bundles, $\xi_1 = (E_1, \pi_1, B, F, G)$ and $\xi_2 = (E_2, \pi_2, B, F, G)$, over the same base space, B, we say that ξ_1 and ξ_2 are *equivalent* if there is an open cover, $\mathcal{U} = (U_\alpha)_{\alpha \in I}$, for B, a family, $\varphi = (\varphi_\alpha)_{\alpha \in I}$, of trivializations, $\varphi_\alpha \colon \pi_1^{-1}(U_\alpha) \to U_\alpha \times F$, for ξ_1, a family, $\varphi' = (\varphi'_\alpha)_{\alpha \in I}$, of trivializations, $\varphi'_\alpha \colon \pi_2^{-1}(U_\alpha) \to U_\alpha \times F$, for ξ_2, and a family, $(\rho_\alpha)_{\alpha \in I}$, of smooth maps, $\rho_\alpha \colon U_\alpha \to G$, such that

$$g'_{\alpha\beta}(b) = \rho_\alpha(b) g_{\alpha\beta}(b) \rho_\beta(b)^{-1}, \qquad \text{for all } b \in U_\alpha \cap U_\beta.$$

Since the trivializations are bijections, the family $(\rho_\alpha)_{\alpha \in I}$ is unique. The following proposition shows that isomorphic fibre bundles are equivalent:

Proposition 7.1 *If two fibre bundles, $\xi_1 = (E_1, \pi_1, B, F, G)$ and $\xi_2 = (E_2, \pi_2, B, F, G)$, over the same base space, B, are isomorphic, then they are equivalent.*

Proof. Let $f \colon \xi_1 \to \xi_2$ be a bundle isomorphism. Then, we know that for some suitable open cover of the base, B, and some trivializing families, (φ_α) for ξ_1 and (φ'_α) for ξ_2, there is a family of maps, $\rho_\alpha \colon U_\alpha \to G$, so that

$$\varphi'_\alpha \circ f \circ {\varphi_\alpha}^{-1}(b, x) = (b, \rho_\alpha(b)(x)),$$

for all $b \in U_\alpha$ and all $x \in F$. Recall that

$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, x) = (b, g_{\alpha\beta}(b)(x)),$$

for all $b \in U_\alpha \cap U_\beta$ and all $x \in F$. This is equivalent to

$$\varphi_\beta^{-1}(b, x) = \varphi_\alpha^{-1}(b, g_{\alpha\beta}(b)(x)),$$

so it is notationally advantageous to introduce ψ_α such that $\psi_\alpha = \varphi_\alpha^{-1}$. Then, we have

$$\psi_\beta(b, x) = \psi_\alpha(b, g_{\alpha\beta}(b)(x))$$

and

$$\varphi'_\alpha \circ f \circ \varphi_\alpha^{-1}(b, x) = (b, \rho_\alpha(b)(x))$$

becomes

$$\psi_\alpha(b, x) = f^{-1} \circ \psi'_\alpha(b, \rho_\alpha(b)(x)).$$

We have

$$\psi_\beta(b, x) = \psi_\alpha(b, g_{\alpha\beta}(b)(x)) = f^{-1} \circ \psi'_\alpha(b, \rho_\alpha(b)(g_{\alpha\beta}(b)(x)))$$

and also

$$\psi_\beta(b, x) = f^{-1} \circ \psi'_\beta(b, \rho_\beta(b)(x)) = f^{-1} \circ \psi'_\alpha(b, g'_{\alpha\beta}(b)(\rho_\beta(b)(x)))$$

from which we deduce

$$\rho_\alpha(b)(g_{\alpha\beta}(b)(x)) = g'_{\alpha\beta}(b)(\rho_\beta(b)(x)),$$

that is

$$g'_{\alpha\beta}(b) = \rho_\alpha(b) g_{\alpha\beta}(b) \rho_\beta(b)^{-1}, \qquad \text{for all } b \in U_\alpha \cap U_\beta,$$

as claimed. $\square$

Remark: If $\xi_1 = (E_1, \pi_1, B_1, F, G)$ and $\xi_2 = (E_2, \pi_2, B_2, F, G)$ are two bundles over different bases and $f \colon \xi_1 \to \xi_2$ is a bundle isomorphism, with $f = (f_B, f_E)$, then f_E and f_B are diffeomorphisms and it is easy to see that we get the conditions

$$g'_{\alpha\beta}(f_B(b)) = \rho_\alpha(b) g_{\alpha\beta}(b) \rho_\beta(b)^{-1}, \qquad \text{for all } b \in U_\alpha \cap U_\beta.$$

The converse of Proposition 7.1 also holds.

Proposition 7.2 *If two fibre bundles, $\xi_1 = (E_1, \pi_1, B, F, G)$ and $\xi_2 = (E_2, \pi_2, B, F, G)$, over the same base space, B, are equivalent then they are isomorphic.*

Proof. Assume that ξ_1 and ξ_2 are equivalent. Then, for some suitable open cover of the base, B, and some trivializing families, (φ_α) for ξ_1 and (φ'_α) for ξ_2, there is a family of maps, $\rho_\alpha \colon U_\alpha \to G$, so that

$$g'_{\alpha\beta}(b) = \rho_\alpha(b) g_{\alpha\beta}(b) \rho_\beta(b)^{-1}, \qquad \text{for all } b \in U_\alpha \cap U_\beta,$$

which can be written as

$$g'_{\alpha\beta}(b) \rho_\beta(b) = \rho_\alpha(b) g_{\alpha\beta}(b).$$

For every U_α, define f_α as the composition

$$\begin{array}{ccccccc}
\pi_1^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times F & \xrightarrow{\widetilde{f}_\alpha} & U_\alpha \times F & \xrightarrow{\varphi'^{\,-1}_\alpha} & \pi_2^{-1}(U_\alpha) \\
\\
z & \longrightarrow & (b, x) & \longrightarrow & (b, \rho_\alpha(b)(x)) & \longrightarrow & \varphi'^{\,-1}_\alpha(b, \rho_\alpha(b)(x)),
\end{array}$$

that is,
$$f_\alpha(z) = {\varphi'_\alpha}^{-1}(b, \rho_\alpha(b)(x)), \qquad \text{with } z \in \pi_1^{-1}(U_\alpha) \text{ and } (b,x) = \varphi_\alpha(z).$$
Clearly, the definition of f_α implies that
$$\varphi'_\alpha \circ f_\alpha \circ {\varphi_\alpha}^{-1}(b,x) = (b, \rho_\alpha(b)(x)),$$
for all $b \in U_\alpha$ and all $x \in F$ and, locally, f_α is a bundle isomorphism with respect to ρ_α. If we can prove that any two f_α and f_β agree on the overlap, $U_\alpha \cap U_\beta$, then the f_α's patch and yield a bundle map between ξ_1 and ξ_2. Now, on $U_\alpha \cap U_\beta$,
$$\varphi_\alpha \circ \varphi_\beta^{-1}(b,x) = (b, g_{\alpha\beta}(b)(x))$$
yields
$$\varphi_\beta^{-1}(b,x) = \varphi_\alpha^{-1}(b, g_{\alpha\beta}(b)(x)).$$
We need to show that for every $z \in U_\alpha \cap U_\beta$,
$$f_\alpha(z) = {\varphi'_\alpha}^{-1}(b, \rho_\alpha(b)(x)) = {\varphi'_\beta}^{-1}(b, \rho_\beta(b)(x')) = f_\beta(z),$$
where $\varphi_\alpha(z) = (b,x)$ and $\varphi_\beta(z) = (b,x')$.

From $z = \varphi_\beta^{-1}(b,x') = \varphi_\alpha^{-1}(b, g_{\alpha\beta}(b)(x'))$, we get
$$x = g_{\alpha\beta}(b)(x').$$
We also have
$${\varphi'_\beta}^{-1}(b, \rho_\beta(b)(x')) = {\varphi'_\alpha}^{-1}(b, g'_{\alpha\beta}(b)(\rho_\beta(b)(x')))$$
and since $g'_{\alpha\beta}(b)\rho_\beta(b) = \rho_\alpha(b)g_{\alpha\beta}(b)$ and $x = g_{\alpha\beta}(b)(x')$ we get
$${\varphi'_\beta}^{-1}(b, \rho_\beta(b)(x')) = {\varphi'_\alpha}^{-1}(b, \rho_\alpha(b)(g_{\alpha\beta}(b))(x')) = {\varphi'_\alpha}^{-1}(b, \rho_\alpha(b)(x)),$$
as desired. Therefore, the f_α's patch to yield a bundle map, f, with respect to the family of maps, $\rho_\alpha \colon U_\alpha \to G$. The map f is bijective because it is an isomorphism on fibres but it remains to show that it is a diffeomorphism. This is a local matter and as the φ_α and φ'_α are diffeomorphisms, it suffices to show that the map, $\widetilde{f_\alpha} \colon U_\alpha \times F \longrightarrow U_\alpha \times F$, given by
$$(b,x) \mapsto (b, \rho_\alpha(b)(x)).$$
is a diffeomorphism. For this, observe that in local coordinates, the Jacobian matrix of this map is of the form
$$J = \begin{pmatrix} I & 0 \\ C & J(\rho_\alpha(b)) \end{pmatrix},$$
where I is the identity matrix and $J(\rho_\alpha(b))$ is the Jacobian matrix of $\rho_\alpha(b)$. Since $\rho_\alpha(b)$ is a diffeomorphism, $\det(J) \neq 0$ and by the Inverse Function Theorem, the map $\widetilde{f_\alpha}$ is a diffeomorphism, as desired. $\square$

Remark: If in Proposition 7.2, $\xi_1 = (E_1, \pi_1, B_1, F, G)$ and $\xi_2 = (E_2, \pi_2, B_2, F, G)$ are two bundles over different bases and if we have a diffeomorphism, $f_B \colon B_1 \to B_2$, and the conditions

$$g'_{\alpha\beta}(f_B(b)) = \rho_\alpha(b) g_{\alpha\beta}(b) \rho_\beta(b)^{-1}, \qquad \text{for all } b \in U_\alpha \cap U_\beta$$

hold, then there is a bundle isomorphism, (f_B, f_E) between ξ_1 and ξ_2.

It follows from Proposition 7.1 and Proposition 7.2 that two bundles over the same base are equivalent iff they are isomorphic, a very useful fact. Actually, we can use the proof of Proposition 7.2 to show that any bundle morphism, $f \colon \xi_1 \to \xi_2$, between two fibre bundles over the same base, B, is a bundle isomorphism. Because a bundle morphism, f, as above is fibre preserving, f is bijective but it is not obvious that its inverse is smooth.

Proposition 7.3 *Any bundle morphism, $f \colon \xi_1 \to \xi_2$, between two fibre bundles over the same base, B, is an isomorphism.*

Proof. Since f is bijective, this is a local matter and it is enough to prove that each, $\widetilde{f}_\alpha \colon U_\alpha \times F \longrightarrow U_\alpha \times F$, is a diffeomorphism, since f can be written as

$$f = {\varphi'_\alpha}^{-1} \circ \widetilde{f}_\alpha \circ \varphi_\alpha,$$

with

$$\widetilde{f}_\alpha(b, x) = (b, \rho_\alpha(b)(x)).$$

However, the end of the proof of Proposition 7.2 shows that $\widetilde{f}_\alpha$ is a diffeomorphism. $\square$

Given a fibre bundle, $\xi = (E, \pi, B, F, G)$, we observed that the family, $g = (g_{\alpha\beta})$, of transition maps, $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$, induced by a trivializing family, $\varphi = (\varphi_\alpha)_{\alpha \in I}$, relative to the open cover, $\mathcal{U} = (U_\alpha)_{\alpha \in I}$, for B satisfies the *cocycle condition*,

$$g_{\alpha\beta}(b) g_{\beta\gamma}(b) = g_{\alpha\gamma}(b),$$

for all α, β, γ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ and all $b \in U_\alpha \cap U_\beta \cap U_\gamma$. Without altering anything, we may assume that $g_{\alpha\beta}$ is the (unique) function from $\emptyset$ to G when $U_\alpha \cap U_\beta = \emptyset$. Then, we call a family, $g = (g_{\alpha\beta})_{(\alpha,\beta) \in I \times I}$, as above a *$\mathcal{U}$-cocycle*, or simply, a *cocycle*. Remarkably, given such a cocycle, g, relative to $\mathcal{U}$, a fibre bundle, ξ_g, over B with fibre, F, and structure group, G, having g as family of transition functions, can be constructed. In view of Proposition 7.1, we say that two cocycles, $g = (g_{\alpha\beta})_{(\alpha,\beta) \in I \times I}$ and $g' = (g_{\alpha\beta})_{(\alpha,\beta) \in I \times I}$, are *equivalent* if there is a family, $(\rho_\alpha)_{\alpha \in I}$, of smooth maps, $\rho_\alpha \colon U_\alpha \to G$, such that

$$g'_{\alpha\beta}(b) = \rho_\alpha(b) g_{\alpha\beta}(b) \rho_\beta(b)^{-1}, \qquad \text{for all } b \in U_\alpha \cap U_\beta.$$

Theorem 7.4 *Given two smooth manifolds, B and F, a Lie group, G, acting effectively on F, an open cover, $\mathcal{U} = (U_\alpha)_{\alpha \in I}$, of B, and a cocycle, $g = (g_{\alpha\beta})_{(\alpha,\beta) \in I \times I}$, there is a fibre bundle, $\xi_g = (E, \pi, B, F, G)$, whose transition maps are the maps in the cocycle, g. Furthermore, if g and g' are equivalent cocycles, then ξ_g and $\xi_{g'}$ are isomorphic.*

Proof sketch. First, we define the space, Z, as the disjoint sum

$$Z = \coprod_{\alpha \in I} U_\alpha \times F.$$

We define the relation, $\simeq$, on $Z \times Z$, as follows: For all $(b, x) \in U_\beta \times F$ and $(b, y) \in U_\alpha \times F$, if $U_\alpha \cap U_\beta \neq \emptyset$,

$$(b, x) \simeq (b, y) \quad \text{iff} \quad y = g_{\alpha\beta}(b)(x).$$

We let $E = Z/\simeq$, and we give E the largest topology such that the injections, $\eta_\alpha \colon U_\alpha \times F \to Z$, are smooth. The cocycle condition insures that $\simeq$ is indeed an equivalence relation. We define $\pi \colon E \to B$ by $\pi([b, x]) = b$. If $p \colon Z \to E$ is the the quotient map, observe that the maps, $p \circ \eta_\alpha \colon U_\alpha \times F \to E$, are injective, and that

$$\pi \circ p \circ \eta_\alpha(b, x) = b.$$

Thus,

$$p \circ \eta_\alpha \colon U_\alpha \times F \to \pi^{-1}(U_\alpha)$$

is a bijection, and we define the trivializing maps by setting

$$\varphi_\alpha = (p \circ \eta_\alpha)^{-1}.$$

It is easily verified that the corresponding transition functions are the original $g_{\alpha\beta}$. There are some details to check. A complete proof (the only one we could find!) is given in Steenrod [141], Part I, Section 3, Theorem 3.2. The fact that ξ_g and $\xi_{g'}$ are equivalent when g and g' are equivalent follows from Proposition 7.2 (see Steenrod [141], Part I, Section 2, Lemma 2.10). $\square$

Remark: (The following paragraph is intended for readers familiar with Čech cohomology.) Obviously, if we start with a fibre bundle, $\xi = (E, \pi, B, F, G)$, whose transition maps are the cocycle, $g = (g_{\alpha\beta})$, and form the fibre bundle, ξ_g, the bundles ξ and ξ_g are equivalent. This leads to a characterization of the set of equivalence classes of fibre bundles over a base space, B, as the *cohomology set*, $\check{H}^1(B, \mathcal{G})$. In the present case, the sheaf, $\mathcal{G}$, is defined such that $\Gamma(U, \mathcal{G})$ is the group of smooth maps from the open subset, U, of B to the Lie group, G. Since G is not abelian, the coboundary maps have to be interpreted multiplicatively. If we define the sets of cochains, $C^k(\mathcal{U}, \mathcal{G})$, so that

$$C^0(\mathcal{U}, \mathcal{G}) = \prod_\alpha \mathcal{G}(U_\alpha), \quad C^1(\mathcal{U}, \mathcal{G}) = \prod_{\alpha<\beta} \mathcal{G}(U_\alpha \cap U_\beta), \quad C^2(\mathcal{U}, \mathcal{G}) = \prod_{\alpha<\beta<\gamma} \mathcal{G}(U_\alpha \cap U_\beta \cap U_\gamma),$$

etc., then it is natural to define,

$$\delta_0 \colon C^0(\mathcal{U}, \mathcal{G}) \to C^1(\mathcal{U}, \mathcal{G}),$$

by

$$(\delta_0 g)_{\alpha\beta} = g_\alpha^{-1} g_\beta,$$

for any $g = (g_\alpha)$, with $g_\alpha \in \Gamma(U_\alpha, \mathcal{G})$. As to

$$\delta_1 \colon C^1(\mathcal{U}, \mathcal{G}) \to C^2(\mathcal{U}, \mathcal{G}),$$

since the cocycle condition in the usual case is

$$g_{\alpha\beta} + g_{\beta\gamma} = g_{\alpha\gamma},$$

we set

$$(\delta_1 g)_{\alpha\beta\gamma} = g_{\alpha\beta} g_{\beta\gamma} g_{\alpha\gamma}^{-1},$$

for any $g = (g_{\alpha\beta})$, with $g_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{G})$. Note that a cocycle, $g = (g_{\alpha\beta})$, is indeed an element of $Z^1(\mathcal{U}, \mathcal{G})$, and the condition for being in the kernel of

$$\delta_1 \colon C^1(\mathcal{U}, \mathcal{G}) \to C^2(\mathcal{U}, \mathcal{G})$$

is the cocycle condition,

$$g_{\alpha\beta}(b) g_{\beta\gamma}(b) = g_{\alpha\gamma}(b),$$

for all $b \in U_\alpha \cap U_\beta \cap U_\gamma$. In the commutative case, two cocycles, g and g', are equivalent if their difference is a boundary, which can be stated as

$$g'_{\alpha\beta} + \rho_\beta = g_{\alpha\beta} + \rho_\alpha = \rho_\alpha + g_{\alpha\beta},$$

where $\rho_\alpha \in \Gamma(U_\alpha, \mathcal{G})$, for all $\alpha \in I$. In the present case, two cocycles, g and g', are equivalent iff there is a family, $(\rho_\alpha)_{\alpha \in I}$, with $\rho_\alpha \in \Gamma(U_\alpha, \mathcal{G})$, such that

$$g'_{\alpha\beta}(b) = \rho_\alpha(b) g_{\alpha\beta}(b) \rho_\beta(b)^{-1},$$

for all $b \in U_\alpha \cap U_\beta$. This is the same condition of equivalence defined earlier. Thus, it is easily seen that if $g, h \in Z^1(\mathcal{U}, \mathcal{G})$, then ξ_g and ξ_h are equivalent iff g and h correspond to the same element of the cohomology set, $\check{H}^1(\mathcal{U}, \mathcal{G})$. As usual, $\check{H}^1(B, \mathcal{G})$ is defined as the direct limit of the directed system of sets, $\check{H}^1(\mathcal{U}, \mathcal{G})$, over the preordered directed family of open covers. For details, see Hirzebruch [77], Section 3.1. In summary, there is a bijection between the equivalence classes of fibre bundles over B (with fibre F and structure group G) and the cohomology set, $\check{H}^1(B, \mathcal{G})$. In the case of line bundles, it turns out that $\check{H}^1(B, \mathcal{G})$ is in fact a group.

As an application of Theorem 7.4, we define the notion of *pullback* (or *induced*) bundle. Say $\xi = (E, \pi, B, F, G)$ is a fibre bundle and assume we have a smooth map, $f \colon N \to B$. We seek a bundle, $f^*\xi$, over N, together with a bundle map, $(f^*, f) \colon f^*\xi \to \xi$,

$$\begin{tikzcd} f^*E \arrow[r, "f^*"] \arrow[d, "\pi^*"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & B \end{tikzcd}$$

where, in fact, f^*E is a pullback in the categorical sense. This means that for any other bundle, ξ', over N and any bundle map,

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & B, \end{tikzcd}$$

there is a unique bundle map, $(\widetilde{f'}, \mathrm{id})\colon \xi' \to f^*\xi$, so that $(f', f) = (f^*, f) \circ (\widetilde{f'}, \mathrm{id})$. Thus, there is an isomorphism (natural),

$$\mathrm{Hom}(\xi', \xi) \cong \mathrm{Hom}(\xi,' f^*\xi).$$

As a consequence, by Proposition 7.3, for any bundle map betwen ξ' and ξ,

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & B, \end{tikzcd}$$

there is an isomorphism, $\xi' \cong f^*\xi$.

The bundle, $f^*\xi$, can be constructed as follows: Pick any open cover, (U_α), of B, then $(f^{-1}(U_\alpha))$ is an open cover of N and check that if $(g_{\alpha\beta})$ is a cocycle for ξ, then the maps, $g_{\alpha\beta} \circ f\colon f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to G$, satisfy the cocycle conditions. Then, $f^*\xi$ is the bundle defined by the cocycle, $(g_{\alpha\beta} \circ f)$. We leave as an exercise to show that the pullback bundle, $f^*\xi$, can be defined explicitly if we set

$$f^*E = \{(n, e) \in N \times E \mid f(n) = \pi(e)\},$$

$\pi^* = pr_1$ and $f^* = pr_2$. For any trivialization, $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$, of ξ we have

$$(\pi^*)^{-1}(f^{-1}(U_\alpha)) = \{(n, e) \in N \times E \mid n \in f^{-1}(U_\alpha), e \in \pi^{-1}(f(n))\},$$

and so, we have a bijection, $\widetilde{\varphi}_\alpha\colon (\pi^*)^{-1}(f^{-1}(U_\alpha)) \to f^{-1}(U_\alpha) \times F$, given by

$$\widetilde{\varphi}_\alpha(n, e) = (n, pr_2(\varphi_\alpha(e))).$$

By giving f^*E the smallest topology that makes each $\widetilde{\varphi}_\alpha$ a diffeomorphism, $\widetilde{\varphi}_\alpha$, is a trivialization of $f^*\xi$ over $f^{-1}(U_\alpha)$ and $f^*\xi$ is a smooth bundle. Note that the fibre of $f^*\xi$ over a point, $n \in N$, is isomorphic to the fibre, $\pi^{-1}(f(n))$, of ξ over $f(n)$. If $g\colon M \to N$ is another smooth map of manifolds, it is easy to check that

$$(f \circ g)^*\xi = g^*(f^*\xi).$$

Given a bundle, $\xi = (E, \pi, B, F, G)$, and a submanifold, N, of B, we define the *restriction of ξ to N* as the bundle, $\xi \restriction N = (\pi^{-1}(N), \pi \restriction \pi^{-1}(N), B, F, G)$.

Experience shows that most objects of interest in geometry (vector fields, differential forms, *etc.*) arise as sections of certain bundles. Furthermore, deciding whether or not a bundle is trivial often reduces to the existence of a (global) section. Thus, we define the important concept of a section right away.

Definition 7.5 Given a fibre bundle, $\xi = (E, \pi, B, F, G)$, a *smooth section* of ξ is a smooth map, $s\colon B \to E$, so that $\pi \circ s = \mathrm{id}_B$. Given an open subset, U, of B, a *(smooth) section of ξ over U* is a smooth map, $s\colon U \to E$, so that $\pi \circ s(b) = b$, for all $b \in U$; we say that s is a *local section* of ξ. The set of all sections over U is denoted $\Gamma(U, \xi)$ and $\Gamma(B, \xi)$ (for short, $\Gamma(\xi)$) is the set of *global sections* of ξ.

Here is an observation that proves useful for constructing global sections. Let $s\colon B \to E$ be a global section of a bundle, ξ. For every trivialization, $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$, let $s_\alpha\colon U_\alpha \to E$ and $\sigma_\alpha\colon U_\alpha \to F$ be given by

$$s_\alpha = s \restriction U_\alpha \quad \text{and} \quad \sigma_\alpha = pr_2 \circ \varphi_\alpha \circ s_\alpha,$$

so that

$$s_\alpha(b) = \varphi_\alpha^{-1}(b, \sigma_\alpha(b)).$$

Obviously, $\pi \circ s_\alpha = \mathrm{id}$, so s_α is a local section of ξ and σ_α is a function, $\sigma_\alpha\colon U_\alpha \to F$. We claim that on overlaps, we have

$$\sigma_\alpha(b) = g_{\alpha\beta}(b)\sigma_\beta(b).$$

Indeed, recall that

$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, x) = (b, g_{\alpha\beta}(b)x),$$

for all $b \in U_\alpha \cap U_\beta$ and all $x \in F$ and as $s_\alpha = s \restriction U_\alpha$ and $s_\beta = s \restriction U_\beta$, s_α and s_β agree on $U_\alpha \cap U_\beta$. Consequently, from

$$s_\alpha(b) = \varphi_\alpha^{-1}(b, \sigma_\alpha(b)) \quad \text{and} \quad s_\beta(b) = \varphi_\beta^{-1}(b, \sigma_\beta(b)),$$

we get

$$\varphi_\alpha^{-1}(b, \sigma_\alpha(b)) = s_\alpha(b) = s_\beta(b) = \varphi_\beta^{-1}(b, \sigma_\beta(b)) = \varphi_\alpha^{-1}(b, g_{\alpha\beta}(b)\sigma_\beta(b)),$$

which implies $\sigma_\alpha(b) = g_{\alpha\beta}(b)\sigma_\beta(b)$, as claimed.

Conversely, assume that we have a collection of functions, $\sigma_\alpha\colon U_\alpha \to F$, satisfying

$$\sigma_\alpha(b) = g_{\alpha\beta}(b)\sigma_\beta(b)$$

on overlaps. Let $s_\alpha\colon U_\alpha \to E$ be given by

$$s_\alpha(b) = \varphi_\alpha^{-1}(b, \sigma_\alpha(b)).$$

Each s_α is a local section and we claim that these sections agree on overlaps, so they patch and define a global section, s. We need to show that

$$s_\alpha(b) = \varphi_\alpha^{-1}(b, \sigma_\alpha(b)) = \varphi_\beta^{-1}(b, \sigma_\beta(b)) = s_\beta(b),$$

for $b \in U_\alpha \cap U_\beta$, that is,

$$(b, \sigma_\alpha(b)) = \varphi_\alpha \circ \varphi_\beta^{-1}(b, \sigma_\beta(b)),$$

and since $\varphi_\alpha \circ \varphi_\beta^{-1}(b, \sigma_\beta(b)) = (b, g_{\alpha\beta}(b)\sigma_\beta(b))$ and by hypothesis, $\sigma_\alpha(b) = g_{\alpha\beta}(b)\sigma_\beta(b)$, our equation $s_\alpha(b) = s_\beta(b)$ is verified.

There are two particularly interesting special cases of fibre bundles:

(1) *Vector bundles*, which are fibre bundles for which the typical fibre is a finite-dimensional vector space, V, and the structure group is a subgroup of the group of linear isomorphisms ($\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, where $n = \dim V$).

(2) *Principal fibre bundles*, which are fibre bundles for which the fibre, F, is equal to the structure group, G, with G acting on itself by left translation.

First, we discuss vector bundles.

7.2 Vector Bundles

Given a real vector space, V, we denote by $\mathrm{GL}(V)$ (or $\mathrm{Aut}(V)$) the vector space of linear invertible maps from V to V. If V has dimension n, then $\mathrm{GL}(V)$ has dimension n^2. Obviously, $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}(n, \mathbb{R})$, so we often write $\mathrm{GL}(n, \mathbb{R})$ instead of $\mathrm{GL}(V)$ but this may be slightly confusing if V is the dual space, W^* of some other space, W. If V is a complex vector space, we also denote by $\mathrm{GL}(V)$ (or $\mathrm{Aut}(V)$) the vector space of linear invertible maps from V to V but this time, $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}(n, \mathbb{C})$, so we often write $\mathrm{GL}(n, \mathbb{C})$ instead of $\mathrm{GL}(V)$.

Definition 7.6 A *rank n real smooth vector bundle with fibre V* is a tuple, $\xi = (E, \pi, B, V)$, such that $(E, \pi, B, V, \mathrm{GL}(V))$ is a smooth fibre bundle, the fibre, V, is a real vector space of dimension n and the following conditions hold:

(a) For every $b \in B$, the fibre, $\pi^{-1}(b)$, is an n-dimensional (real) vector space.

(b) For every trivialization, $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$, for every $b \in U_\alpha$, the restriction of φ_α to the fibre, $\pi^{-1}(b)$, is a linear isomorphism, $\pi^{-1}(b) \longrightarrow V$.

A *rank n complex smooth vector bundle with fibre V* is a tuple, $\xi = (E, \pi, B, V)$, such that $(E, \pi, B, V, \mathrm{GL}(V))$ is a smooth fibre bundle such that the fibre, V, is an n-dimensional complex vector space (viewed as a real smooth manifold) and conditions (a) and (b) above hold (for complex vector spaces). When $n = 1$, a vector bundle is called a *line bundle*.

The trivial vector bundle, $E = B \times V$, is often denoted ϵ^V. When $V = \mathbb{R}^k$, we also use the notation ϵ^k. Given a (smooth) manifold, M, of dimension n, the tangent bundle, TM, and the cotangent bundle, T^*M, are rank n vector bundles. Indeed, in Section 3.3, we defined trivialization maps (denoted τ_U) for TM. Let us compute the transition functions for the tangent bundle, TM, where M is a smooth manifold of dimension n. Recall from Definition 3.12 that for every $p \in M$, the tangent space, T_pM, consists of all equivalence classes of triples, (U, φ, x), where (U, φ) is a chart with $p \in U$, $x \in \mathbb{R}^n$, and the equivalence relation on triples is given by

$$(U, \varphi, x) \equiv (V, \psi, y) \quad \text{iff} \quad (\psi \circ \varphi^{-1})'_{\varphi(p)}(x) = y.$$

We have a natural isomorphism, $\theta_{U,\varphi,p} \colon \mathbb{R}^n \to T_pM$, between $\mathbb{R}^n$ and T_pM given by

$$\theta_{U,\varphi,p} \colon x \mapsto [(U, \varphi, x)], \qquad x \in \mathbb{R}^n.$$

Observe that for any two overlapping charts, (U, φ) and (V, ψ),

$$\theta^{-1}_{V,\psi,p} \circ \theta_{U,\varphi,p} = (\psi \circ \varphi^{-1})'_{\varphi(p)}.$$

We let TM be the disjoint union,

$$TM = \bigcup_{p \in M} T_pM,$$

define the projection, $\pi \colon TM \to M$, so that $\pi(v) = p$ if $v \in T_pM$, and we give TM the weakest topology that makes the functions, $\widetilde{\varphi} \colon \pi^{-1}(U) \to \mathbb{R}^{2n}$, given by

$$\widetilde{\varphi}(v) = (\varphi \circ \pi(v), \theta^{-1}_{U,\varphi,\pi(v)}(v)),$$

continuous, where (U, φ) is any chart of M. Each function, $\widetilde{\varphi} \colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$ is a homeomorphism and given any two overlapping charts, (U, φ) and (V, ψ), as $\theta^{-1}_{V,\psi,p} \circ \theta_{U,\varphi,p} = (\psi \circ \varphi^{-1})'_{\varphi(p)}$, the transition map,

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1} \colon \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n,$$

is given by

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}(z, x) = (\psi \circ \varphi^{-1}(z), (\psi \circ \varphi^{-1})'_z(x)), \qquad (z, x) \in \varphi(U \cap V) \times \mathbb{R}^n.$$

It is clear that $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$ is smooth. Moreover, the bijection,

$$\tau_U \colon \pi^{-1}(U) \to U \times \mathbb{R}^n,$$

given by

$$\tau_U(v) = (\pi(v), \theta^{-1}_{U,\varphi,\pi(v)}(v))$$

satisfies $pr_1 \circ \tau_U = \pi$ on $\pi^{-1}(U)$, is a linear isomorphism restricted to fibres and so, it is a trivialization for TM. For any two overlapping charts, $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), the transition function, $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$, is given by

$$g_{\alpha\beta}(p)(x) = (\varphi_\alpha \circ \varphi_\beta^{-1})'_{\varphi(p)}(x).$$

We can also compute trivialization maps for T^*M. This time, T^*M is the disjoint union,

$$T^*M = \bigcup_{p \in M} T_p^*M,$$

and $\pi\colon T^*M \to M$ is given by $\pi(\omega) = p$ if $\omega \in T_p^*M$, where T_p^*M is the dual of the tangent space, T_pM. For each chart, (U, φ), by dualizing the map, $\theta_{U,\varphi,p}\colon \mathbb{R}^n \to T_p(M)$, we obtain an isomorphism, $\theta_{U,\varphi,p}^\top\colon T_p^*M \to (\mathbb{R}^n)^*$. Composing $\theta_{U,\varphi,p}^\top$ with the isomorphism, $\iota\colon (\mathbb{R}^n)^* \to \mathbb{R}^n$ (induced by the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ and its dual basis), we get an isomorphism, $\theta_{U,\varphi,p}^* = \iota \circ \theta_{U,\varphi,p}^\top\colon T_p^*M \to \mathbb{R}^n$. Then, define the bijection,

$$\widetilde{\varphi}^*\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n},$$

by

$$\widetilde{\varphi}^*(\omega) = (\varphi \circ \pi(\omega), \theta_{U,\varphi,\pi(\omega)}^*(\omega)),$$

with $\omega \in \pi^{-1}(U)$. We give T^*M the weakest topology that makes the functions $\widetilde{\varphi}^*$ continuous and then each function, $\widetilde{\varphi}^*$, is a homeomorphism. Given any two overlapping charts, (U, φ) and (V, ψ), as

$$\theta_{V,\psi,p}^{-1} \circ \theta_{U,\varphi,p} = (\psi \circ \varphi^{-1})'_{\varphi(p)},$$

by dualization we get

$$\theta_{U,\varphi,p}^\top \circ (\theta_{V,\psi,p}^\top)^{-1} = \theta_{U,\varphi,p}^\top \circ (\theta_{V,\psi,p}^{-1})^\top = ((\psi \circ \varphi^{-1})'_{\varphi(p)})^\top,$$

then

$$\theta_{V,\psi,p}^\top \circ (\theta_{U,\varphi,p}^\top)^{-1} = (((\psi \circ \varphi^{-1})'_{\varphi(p)})^\top)^{-1},$$

and so

$$\iota \circ \theta_{V,\psi,p}^\top \circ (\theta_{U,\varphi,p}^\top)^{-1} \circ \iota^{-1} = \iota \circ (((\psi \circ \varphi^{-1})'_{\varphi(p)})^\top)^{-1} \circ \iota^{-1},$$

that is,

$$\theta_{V,\psi,p}^* \circ (\theta_{U,\varphi,p}^*)^{-1} = \iota \circ (((\psi \circ \varphi^{-1})'_{\varphi(p)})^\top)^{-1} \circ \iota^{-1}.$$

Consequently, the transition map,

$$\widetilde{\psi}^* \circ (\widetilde{\varphi}^*)^{-1}\colon \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n,$$

is given by

$$\widetilde{\psi}^* \circ (\widetilde{\varphi}^*)^{-1}(z, x) = (\psi \circ \varphi^{-1}(z), \iota \circ (((\psi \circ \varphi^{-1})'_z)^\top)^{-1} \circ \iota^{-1}(x)), \qquad (z, x) \in \varphi(U \cap V) \times \mathbb{R}^n.$$

If we view $(\psi \circ \varphi^{-1})'_z$ as a matrix, then we can forget ι and the second component of $\widetilde{\psi}^* \circ (\widetilde{\varphi}^*)^{-1}(z, x)$ is $(((\psi \circ \varphi^{-1})'_z)^\top)^{-1} x$.

We also have trivialization maps, $\tau^*_U \colon \pi^{-1}(U) \to U \times (\mathbb{R}^n)^*$, for T^*M given by

$$\tau^*_U(\omega) = (\pi(\omega), \theta^\top_{U,\varphi,\pi(\omega)}(\omega)),$$

for all $\omega \in \pi^{-1}(U)$. The transition function, $g^*_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$, is given by

$$\begin{aligned}
g^*_{\alpha\beta}(p)(\eta) &= \tau^*_{U_\alpha,p} \circ (\tau^*_{U_\beta,p})^{-1}(\eta) \\
&= \theta^\top_{U_\alpha,\varphi_\alpha,\pi(\eta)} \circ (\theta^\top_{U_\beta,\varphi_\beta,\pi(\eta)})^{-1}(\eta) \\
&= ((\theta^{-1}_{U_\alpha,\varphi_\alpha,\pi(\eta)} \circ \theta_{U_\beta,\varphi_\beta,\pi(\eta)})^\top)^{-1}(\eta) \\
&= (((\varphi_\alpha \circ \varphi_\beta^{-1})'_{\varphi(p)})^\top)^{-1}(\eta),
\end{aligned}$$

with $\eta \in (\mathbb{R}^n)^*$. Also note that $\mathrm{GL}(n, \mathbb{R})$ should really be $\mathrm{GL}((\mathbb{R}^n)^*)$, but $\mathrm{GL}((\mathbb{R}^n)^*)$ is isomorphic to $\mathrm{GL}(n, \mathbb{R})$. We conclude that

$$g^*_{\alpha\beta}(p) = (g_{\alpha\beta}(p)^\top)^{-1}, \qquad \text{for every } p \in M.$$

This is a general property of dual bundles, see Property (f) in Section 7.3.

Maps of vector bundles are maps of fibre bundles such that the isomorphisms between fibres are linear.

Definition 7.7 Given two vector bundles, $\xi_1 = (E_1, \pi_1, B_1, V)$ and $\xi_2 = (E_2, \pi_2, B_2, V)$, with the same typical fibre, V, a *bundle map (or bundle morphism)*, $f \colon \xi_1 \to \xi_2$, is a pair, $f = (f_E, f_B)$, of smooth maps, $f_E \colon E_1 \to E_2$ and $f_B \colon B_1 \to B_2$, such that

(a) The following diagram commutes:

$$\begin{tikzcd}
E_1 \arrow[r, "f_E"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\
B_1 \arrow[r, "f_B"'] & B_2
\end{tikzcd}$$

(b) For every $b \in B_1$, the map of fibres,

$$f_E \restriction \pi_1^{-1}(b) \colon \pi_1^{-1}(b) \to \pi_2^{-1}(f_B(b)),$$

is a bijective linear map.

A bundle map *isomorphism*, $f \colon \xi_1 \to \xi_2$, is defined as in Definition 7.2. Given two vector bundles, $\xi_1 = (E_1, \pi_1, B, V)$ and $\xi_2 = (E_2, \pi_2, B, V)$, over the same base space, B, we require $f_B = \mathrm{id}$.

Remark: Some authors do not require the preservation of fibres, that is, the map

$$f_E \restriction \pi_1^{-1}(b)\colon \pi_1^{-1}(b) \to \pi_2^{-1}(f_B(b))$$

is simply a linear map. It is automatically bijective for bundle isomorphisms.

Note that Definition 7.7 does not include condition (b) of Definition 7.3. However, because the restrictions of the maps φ_α, φ'_β and f to the fibres are linear isomorphisms, it turns out that condition (b) (of Definition 7.3) does hold. Indeed, if $f_B(U_\alpha) \subseteq V_\beta$, then

$$\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}\colon U_\alpha \times V \longrightarrow V_\beta \times V$$

is a smooth map and, for every $b \in B$, its restriction to $\{b\} \times V$ is a linear isomorphism between $\{b\} \times V$ and $\{f_B(b)\} \times V$. Therefore, there is a smooth map, $\rho_\alpha\colon U_\alpha \to \mathrm{GL}(n, \mathbb{R})$, so that

$$\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}(b, x) = (f_B(b), \rho_\alpha(b)(x))$$

and a vector bundle map is a fibre bundle map.

A *holomorphic vector bundle* is a fibre bundle where E, B are complex manifolds, V is a complex vector space of dimension n, the map π is holomorphic, the φ_α are biholomorphic, and the transition functions, $g_{\alpha\beta}$, are holomorphic. When $n = 1$, a holomorphic vector bundle is called a *holomorphic line bundle*.

Definition 7.4 also applies to vector bundles (just replace G by $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$) and defines the notion of equivalence of vector bundles over B. Since vector bundle maps are fibre bundle maps, Propositions 7.1 and 7.2 immediately yield

Proposition 7.5 *Two vector bundles, $\xi_1 = (E_1, \pi_1, B, V)$ and $\xi_2 = (E_2, \pi_2, B, V)$, over the same base space, B, are equivalent iff they are isomorphic.*

Since a vector bundle map is a fibre bundle map, Proposition 7.3 also yields the useful fact:

Proposition 7.6 *Any vector bundle map, $f\colon \xi_1 \to \xi_2$, between two vector bundles over the same base, B, is an isomorphism.*

Theorem 7.4 also holds for vector bundles and yields a technique for constructing new vector bundles over some base, B.

Theorem 7.7 *Given a smooth manifold, B, an n-dimensional (real, resp. complex) vector space, V, an open cover, $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of B, and a cocycle, $g = (g_{\alpha\beta})_{(\alpha,\beta) \in I \times I}$ (with $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$, resp. $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$), there is a vector bundle, $\xi_g = (E, \pi, B, V)$, whose transition maps are the maps in the cocycle, g. Furthermore, if g and g' are equivalent cocycles, then ξ_g and $\xi_{g'}$ are equivalent.*

Observe that a coycle, $g = (g_{\alpha\beta})_{(\alpha,\beta)\in I\times I}$, is given by a family of matrices in $\mathrm{GL}(n, \mathbb{R})$ (resp. $\mathrm{GL}(n, \mathbb{C})$).

A vector bundle, ξ, always has a global section, namely the *zero section*, which assigns the element $0 \in \pi^{-1}(b)$, to every $b \in B$. A global section, s, is a *non-zero section* iff $s(b) \neq 0$ for all $b \in B$. It is usually difficult to decide whether a bundle has a nonzero section. This question is related to the nontriviality of the bundle and there is a useful test for triviality. Assume ξ is a trivial rank n vector bundle. Then, there is a bundle isomorphism, $f\colon B \times V \to \xi$. For every $b \in B$, we know that $f(b, -)$ is a linear isomorphism, so for any choice of a basis, $(e_1, \ldots, e_n)$ of V, we get a basis, $(f(b, e_1), \ldots, f(b, e_n))$, of the fibre, $\pi^{-1}(b)$. Thus, we have n global sections, $s_1 = f(-, e_1), \ldots, s_n = f(-, e_n)$, such that $(s_1(b), \ldots, s_n(b))$ forms a basis of the fibre, $\pi^{-1}(b)$, for every $b \in B$.

Definition 7.8 Let $\xi = (E, \pi, B, V)$ be a rank n vector bundle. For any open subset, $U \subseteq B$, an n-tuple of local sections, $(s_1, \ldots, s_n)$, over U if called a *frame over* U iff $(s_1(b), \ldots, s_n(b))$ is a basis of the fibre, $\pi^{-1}(b)$, for every $b \in U$. If $U = B$, then the s_i are global sections and $(s_1, \ldots, s_n)$ is called a *frame* (of ξ).

The notion of a frame is due to Élie Cartan who (after Darboux) made extensive use of them under the name of *moving frame* (and the *moving frame method*). Cartan's terminology is intuitively clear: As a point, b, moves in U, the frame, $(s_1(b), \ldots, s_n(b))$, moves from fibre to fibre. Physicists refer to a frame as a choice of *local gauge*.

The converse of the property established just before Definition 7.8 is also true.

Proposition 7.8 *A rank n vector bundle, ξ, is trivial iff it possesses a frame of global sections.*

Proof. We only need to prove that if ξ has a frame, $(s_1, \ldots, s_n)$, then it is trivial. Pick a basis, $(e_1, \ldots, e_n)$, of V and define the map, $f\colon B \times V \to \xi$, as follows:

$$f(b, v) = \sum_{i=1}^{n} v_i s_i(b),$$

where $v = \sum_{i=1}^{n} v_i e_i$. Clearly, f is bijective on fibres, smooth, and a map of vector bundles. By Proposition 7.6, the bundle map, f, is an isomorphism. $\square$

As an illustration of Proposition 7.8 we can prove that the tangent bundle, TS^1, of the circle, is trivial. Indeed, we can find a section that is everywhere nonzero, *i.e.* a non-vanishing vector field, namely

$$s(\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta).$$

The reader should try proving that TS^3 is also trivial (use the quaternions). However, TS^2 is nontrivial, although this not so easy to prove. More generally, it can be shown that TS^n is nontrivial for all even $n \geq 2$. It can even be shown that S^1, S^3 and S^7 are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.

Remark: A manifold, M, such that its tangent bundle, TM, is trivial is called *parallelizable*.

The above considerations show that if ξ is any rank n vector bundle, not necessarily trivial, then for any local trivialization, $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$, there are always frames over U_α. Indeed, for every choice of a basis, $(e_1, \ldots, e_n)$, of the typical fibre, V, if we set

$$s_i^\alpha(b) = \varphi_\alpha^{-1}(b, e_i), \qquad b \in U_\alpha,\ 1 \leq i \leq n,$$

then $(s_1^\alpha, \ldots, s_n^\alpha)$ is a frame over U_α.

Given any two vector spaces, V and W, both of dimension n, we denote by $\mathrm{Iso}(V, W)$ the space of all linear isomorphisms between V and W. The *space of n-frames*, $F(V)$, is the set of bases of V. Since every basis, $(v_1, \ldots, v_n)$, of V is in one-to-one correspondence with the map from $\mathbb{R}^n$ to V given by $e_i \mapsto v_i$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$ (so, $e_i = (0, \ldots, 1, \ldots 0)$ with the 1 in the ith slot), we have an isomorphism,

$$F(V) \cong \mathrm{Iso}(\mathbb{R}^n, V).$$

(The choice of a basis in V also yields an isomorphism, $\mathrm{Iso}(\mathbb{R}^n, V) \cong \mathrm{GL}(n, \mathbb{R})$, so $F(V) \cong \mathrm{GL}(n, \mathbb{R})$.)

For any rank n vector bundle, ξ, we can form the *frame bundle*, $F(\xi)$, by replacing the fibre, $\pi^{-1}(b)$, over any $b \in B$ by $F(\pi^{-1}(b))$. In fact, $F(\xi)$ can be constructed using Theorem 7.4. Indeed, identifying $F(V)$ with $\mathrm{Iso}(\mathbb{R}^n, V)$, the group $\mathrm{GL}(n, \mathbb{R})$ acts on $F(V)$ effectively on the left *via*

$$A \cdot v = v \circ A^{-1}.$$

(The only reason for using A^{-1} instead of A is that we want a left action.) The resulting bundle has typical fibre, $F(V) \cong \mathrm{GL}(n, \mathbb{R})$, and turns out to be a principal bundle. We will take a closer look at principal bundles in Section 7.5.

We conclude this section with an example of a bundle that plays an important role in algebraic geometry, the *canonical line bundle on* $\mathbb{RP}^n$. Let $H_n^{\mathbb{R}} \subseteq \mathbb{RP}^n \times \mathbb{R}^{n+1}$ be the subset,

$$H_n^{\mathbb{R}} = \{(L, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in L\},$$

where $\mathbb{RP}^n$ is viewed as the set of lines, L, in $\mathbb{R}^{n+1}$ through 0, or more explicitly,

$$H_n^{\mathbb{R}} = \{((x_0\colon \cdots : x_n), \lambda(x_0, \ldots, x_n)) \mid (x_0\colon \cdots : x_n) \in \mathbb{RP}^n,\ \lambda \in \mathbb{R}\}.$$

Geometrically, $H_n^{\mathbb{R}}$ consists of the set of lines, $[(x_0, \ldots, x_n)]$, associated with points, $(x_0\colon \cdots : x_n)$, of $\mathbb{RP}^n$. If we consider the projection, $\pi\colon H_n^{\mathbb{R}} \to \mathbb{RP}^n$, of $H_n^{\mathbb{R}}$ onto $\mathbb{RP}^n$, we see that each fibre is isomorphic to $\mathbb{R}$. We claim that $H_n^{\mathbb{R}}$ is a line bundle. For this, we exhibit trivializations, leaving as an exercise the fact that $H_n^{\mathbb{R}}$ is a manifold.

Recall the open cover, $U_0, \ldots, U_n$, of $\mathbb{RP}^n$, where

$$U_i = \{(x_0\colon \cdots : x_n) \in \mathbb{RP}^n \mid x_i \neq 0\}.$$

Then, the maps, $\varphi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}$, given by

$$\varphi_i((x_0\colon \cdots : x_n), \lambda(x_0, \ldots, x_n)) = ((x_0\colon \cdots : x_n), \lambda x_i)$$

are trivializations. The transition function, $g_{ij}\colon U_i \cap U_j \to \mathrm{GL}(1, \mathbb{R})$, is given by

$$g_{ij}(x_0\colon \cdots : x_n)(u) = \frac{x_i}{x_j}\, u,$$

where we identify $\mathrm{GL}(1, \mathbb{R})$ and $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Interestingly, the bundle $H_n^{\mathbb{R}}$ is nontrivial for all $n \geq 1$. For this, by Proposition 7.8 and since $H_n^{\mathbb{R}}$ is a line bundle, it suffices to prove that every global section vanishes at some point. So, let σ be any section of $H_n^{\mathbb{R}}$. Composing the projection, $p\colon S^n \longrightarrow \mathbb{RP}^n$, with σ, we get a smooth function, $s = \sigma \circ p\colon S^n \longrightarrow H_n^{\mathbb{R}}$, and we have

$$s(x) = (p(x), f(x)x),$$

for every $x \in S^n$, where $f\colon S^n \to \mathbb{R}$ is a smooth function. Moreover, f satisfies

$$f(-x) = -f(x),$$

since $s(-x) = s(x)$. As S^n is connected and f is continuous, by the intermediate value theorem, there is some x such that $f(x) = 0$, and thus, σ vanishes, as desired.

The reader should look for a geometric representation of $H_1^{\mathbb{R}}$. It turns out that $H_1^{\mathbb{R}}$ is an open Möbius strip, that is, a Möbius strip with its boundary deleted (see Milnor and Stasheff [110], Chapter 2). There is also a complex version of the canonical line bundle on $\mathbb{CP}^n$, with

$$H_n = \{(L, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \in L\},$$

where $\mathbb{CP}^n$ is viewed as the set of lines, L, in $\mathbb{C}^{n+1}$ through 0. These bundles are also nontrivial. Furthermore, unlike the real case, the dual bundle, H_n^*, is not isomorphic to H_n. Indeed, H_n^* turns out to have nonzero global holomorphic sections!

7.3 Operations on Vector Bundles

Because the fibres of a vector bundle are vector spaces all isomorphic to some given space, V, we can perform operations on vector bundles that extend familiar operations on vector spaces, such as: direct sum, tensor product, (linear) function space, and dual space. Basically, the same operation is applied on fibres. It is usually more convenient to define operations on vector bundles in terms of operations on cocycles, using Theorem 7.7.

(a) (*Whitney Sum* or *Direct Sum*)

If $\xi = (E, \pi, B, V)$ is a rank m vector bundle and $\xi' = (E', \pi', B, W)$ is a rank n vector bundle, both over the same base, B, then their Whitney sum, $\xi \oplus \xi'$, is the rank $(m+n)$ vector bundle whose fibre over any $b \in B$ is the direct sum, $E_b \oplus E'_b$, that is, the vector bundle with typical fibre $V \oplus W$ (given by Theorem 7.7) specified by the cocycle whose matrices are

$$\begin{pmatrix} g_{\alpha\beta}(b) & 0 \\ 0 & g'_{\alpha\beta}(b) \end{pmatrix}, \qquad b \in U_\alpha \cap U_\beta.$$

(b) (*Tensor Product*)

If $\xi = (E, \pi, B, V)$ is a rank m vector bundle and $\xi' = (E', \pi', B, W)$ is a rank n vector bundle, both over the same base, B, then their tensor product, $\xi \otimes \xi'$, is the rank mn vector bundle whose fibre over any $b \in B$ is the tensor product, $E_b \otimes E'_b$, that is, the vector bundle with typical fibre $V \otimes W$ (given by Theorem 7.7) specified by the cocycle whose matrices are

$$g_{\alpha\beta}(b) \otimes g'_{\alpha\beta}(b), \qquad b \in U_\alpha \cap U_\beta.$$

(Here, we identify a matrix with the corresponding linear map.)

(c) (*Tensor Power*)

If $\xi = (E, \pi, B, V)$ is a rank m vector bundle, then for any $k \geq 0$, we can define the tensor power bundle, $\xi^{\otimes k}$, whose fibre over any $b \in \xi$ is the tensor power, $E_b^{\otimes k}$ and with typical fibre $V^{\otimes k}$. (When $k = 0$, the fibre is $\mathbb{R}$ or $\mathbb{C}$). The bundle $\xi^{\otimes k}$ is determined by the cocycle

$$g_{\alpha\beta}^{\otimes k}(b), \qquad b \in U_\alpha \cap U_\beta.$$

(d) (*Exterior Power*)

If $\xi = (E, \pi, B, V)$ is a rank m vector bundle, then for any $k \geq 0$, we can define the exterior power bundle, $\bigwedge^k \xi$, whose fibre over any $b \in \xi$ is the exterior power, $\bigwedge^k E_b$ and with typical fibre $\bigwedge^k V$. The bundle $\bigwedge^k \xi$ is determined by the cocycle

$$\bigwedge^k g_{\alpha\beta}(b), \qquad b \in U_\alpha \cap U_\beta.$$

Using (a), we also have the *exterior algebra bundle*, $\bigwedge \xi = \bigoplus_{k=0}^m \bigwedge^k \xi$. (When $k = 0$, the fibre is $\mathbb{R}$ or $\mathbb{C}$).

(e) (*Symmetric Power*) If $\xi = (E, \pi, B, V)$ is a rank m vector bundle, then for any $k \geq 0$, we can define the symmetric power bundle, $\mathrm{Sym}^k\, \xi$, whose fibre over any $b \in \xi$ is the exterior power, $\mathrm{Sym}^k\, E_b$ and with typical fibre $\mathrm{Sym}^k\, V$. (When $k = 0$, the fibre is $\mathbb{R}$ or $\mathbb{C}$). The bundle $\mathrm{Sym}^k \xi$ is determined by the cocycle

$$\mathrm{Sym}^k\, g_{\alpha\beta}(b), \qquad b \in U_\alpha \cap U_\beta.$$

(f) (*Dual Bundle*) If $\xi = (E, \pi, B, V)$ is a rank m vector bundle, then its dual bundle, ξ^*, is the rank m vector bundle whose fibre over any $b \in B$ is the dual space, E_b^*, that is, the vector bundle with typical fibre V^* (given by Theorem 7.7) specified by the cocycle whose matrices are

$$(g_{\alpha\beta}(b)^\top)^{-1}, \qquad b \in U_\alpha \cap U_\beta.$$

The reason for this seemingly complicated formula is this: For any trivialization, $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$, for any $b \in B$, recall that the restriction, $\varphi_{\alpha,b} \colon \pi^{-1}(b) \to V$, of φ_α to $\pi^{-1}(b)$ is a linear isomorphism. By dualization we get a map, $\varphi_{\alpha,b}^\top \colon V^* \to (\pi^{-1}(b))^*$, and thus, $\varphi_{\alpha,b}^*$ for ξ^* is given by

$$\varphi_{\alpha,b}^* = (\varphi_{\alpha,b}^\top)^{-1} \colon (\pi^{-1}(b))^* \to V^*.$$

As $g_{\alpha\beta}^*(b) = \varphi_{\alpha,b}^* \circ (\varphi_{\beta,b}^*)^{-1}$, we get

$$\begin{aligned}
g_{\alpha\beta}^*(b) &= (\varphi_{\alpha,b}^\top)^{-1} \circ \varphi_{\beta,b}^\top \\
&= ((\varphi_{\beta,b}^\top)^{-1} \circ \varphi_{\alpha,b}^\top)^{-1} \\
&= (\varphi_{\beta,b}^{-1})^\top \circ \varphi_{\alpha,b}^\top)^{-1} \\
&= ((\varphi_{\alpha,b} \circ \varphi_{\beta,b}^{-1})^\top)^{-1} \\
&= (g_{\alpha\beta}(b)^\top)^{-1},
\end{aligned}$$

as claimed.

(g) ($\mathcal{H}om$ *Bundle*)

If $\xi = (E, \pi, B, V)$ is a rank m vector bundle and $\xi' = (E', \pi', B, W)$ is a rank n vector bundle, both over the same base, B, then their $\mathcal{H}om$ bundle, $\mathcal{H}om(\xi, \xi')$, is the rank mn vector bundle whose fibre over any $b \in B$ is $\mathrm{Hom}(E_b, E'_b)$, that is, the vector bundle with typical fibre $\mathrm{Hom}(V, W)$. The transition functions of this bundle are obtained as follows: For any trivializations, $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$ and $\varphi'_\alpha \colon (\pi')^{-1}(U_\alpha) \to U_\alpha \times W$, for any $b \in B$, recall that the restrictions, $\varphi_{\alpha,b} \colon \pi^{-1}(b) \to V$ and $\varphi'_{\alpha,b} \colon (\pi')^{-1}(b) \to W$ are linear isomorphisms. Then, we have a linear isomorphism, $\varphi_{\alpha,b}^{\mathrm{Hom}} \colon \mathrm{Hom}(\pi^{-1}(b), (\pi')^{-1}(b)) \longrightarrow \mathrm{Hom}(V, W)$, given by

$$\varphi_{\alpha,b}^{\mathrm{Hom}}(f) = \varphi'_{\alpha,b} \circ f \circ \varphi_{\alpha,b}^{-1}, \qquad f \in \mathrm{Hom}(\pi^{-1}(b), (\pi')^{-1}(b)).$$

Then, $g_{\alpha\beta}^{\mathrm{Hom}}(b) = \varphi_{\alpha,b}^{\mathrm{Hom}} \circ (\varphi_{\beta,b}^{\mathrm{Hom}})^{-1}$.

(h) (*Tensor Bundle of type* (r, s))

If $\xi = (E, \pi, B, V)$ is a rank m vector bundle, then for any $r, s \geq 0$, we can define the bundle, $T^{r,s}\,\xi$, whose fibre over any $b \in \xi$ is the tensor space $T^{r,s}\,E_b$ and with typical fibre $T^{r,s}\,V$. The bundle $T^{r,s}\xi$ is determined by the cocycle

$$g_{\alpha\beta}^{\otimes r}(b) \otimes ((g_{\alpha\beta}(b)^\top)^{-1})^{\otimes s}(b), \qquad b \in U_\alpha \cap U_\beta.$$

In view of the canonical isomorphism, $\mathrm{Hom}(V,W) \cong V^* \otimes W$, it is easy to show that $\mathcal{H}om(\xi, \xi')$, is isomorphic to $\xi^* \otimes \xi'$. Similarly, ξ^{**} is isomorphic to ξ. We also have the isomorphism

$$T^{r,s}\xi \cong \xi^{\otimes r} \otimes (\xi^*)^{\otimes s}.$$

Do not confuse the space of bundle morphisms, $\mathrm{Hom}(\xi, \xi')$, with the $\mathcal{H}om$ bundle, $\mathcal{H}om(\xi, \xi')$. However, observe that $\mathrm{Hom}(\xi, \xi')$ is the set of global sections of $\mathcal{H}om(\xi, \xi')$.

As an illustration of (d), consider the exterior power, $\bigwedge^r T^*M$, where M is a manifold of dimension n. We have trivialization maps, $\tau_U^*\colon \pi^{-1}(U) \to U \times \bigwedge^r(\mathbb{R}^n)^*$, for $\bigwedge^r T^*M$ given by

$$\tau_U^*(\omega) = (\pi(\omega), \bigwedge^r \theta_{U,\varphi,\pi(\omega)}^\top(\omega)),$$

for all $\omega \in \pi^{-1}(U)$. The transition function, $g_{\alpha\beta}^{\bigwedge^r}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$, is given by

$$g_{\alpha\beta}^{\bigwedge^r}(p)(\omega) = (\bigwedge^r (((\varphi_\alpha \circ \varphi_\beta^{-1})'_{\varphi(p)})^\top)^{-1})(\omega),$$

for all $\omega \in \pi^{-1}(U)$. Consequently,

$$g_{\alpha\beta}^{\bigwedge^r}(p) = \bigwedge^r (g_{\alpha\beta}(p)^\top)^{-1},$$

for every $p \in M$, a special case of (h).

For rank 1 vector bundles, that is, line bundles, it is easy to show that the set of equivalence classes of line bundles over a base, B, forms a group, where the group operation is $\otimes$, the inverse is $*$ (dual) and the identity element is the trivial bundle. This is the *Picard group* of B.

In general, the dual, E^*, of a bundle is *not* isomorphic to the original bundle, E. This is because, V^* is *not* canonically isomorphic to V and to get a bundle isomorphism between ξ and ξ^*, we need canonical isomorphisms between the fibres. However, if ξ is real, then (using a partition of unity) ξ can be given a Euclidean metric and so, ξ and ξ^* are isomorphic.

It is *not* true in general that a complex vector bundle is isomorphic to its dual because a Hermitian metric only induces a canonical isomorphism between E^* and $\overline{E}$, where $\overline{E}$ is the conjugate of E, with scalar multiplication in $\overline{E}$ given by $(z, w) \mapsto \overline{w}z$.

Remark: Given a real vector bundle, ξ, the *complexification, $\xi_{\mathbb{C}}$, of* ξ is the complex vector bundle defined by

$$\xi_{\mathbb{C}} = \xi \otimes_{\mathbb{R}} \epsilon_{\mathbb{C}},$$

where $\epsilon_{\mathbb{C}} = B \times \mathbb{C}$ is the trivial complex line bundle. Given a complex vector bundle, ξ, by viewing its fibre as a real vector space we obtain the real vector bundle, $\xi_{\mathbb{R}}$. The following facts can be shown:

(1) For every real vector bundle, ξ,

$$(\xi_{\mathbb{C}})_{\mathbb{R}} \cong \xi \oplus \xi.$$

(2) For every complex vector bundle, ξ,

$$(\xi_{\mathbb{R}})_{\mathbb{C}} \cong \xi \oplus \xi^*.$$

The notion of subbundle is defined as follows:

Definition 7.9 Given two vector bundles, $\xi = (E, \pi, B, V)$ and $\xi' = (E', \pi', B, V')$, over the same base, B, we say that ξ is a *subbundle* of ξ' iff E is a submanifold of E', V is a subspace of V' and for every $b \in B$, the fibre, $\pi^{-1}(b)$, is a subspace of the fibre, $(\pi')^{-1}(b)$.

If ξ is a subbundle of ξ', we can form the *quotient bundle*, ξ'/ξ, as the bundle over B whose fibre at $b \in B$ is the quotient space $(\pi')^{-1}(b)/\pi^{-1}(b)$. We leave it as an exercise to define trivializations for ξ'/ξ. In particular, if N is a submanifold of M, then TN is a subbundle of $TM \upharpoonright N$ and the quotient bundle $(TM \upharpoonright N)/TN$ is called the *normal bundle* of N in M.

7.4 Metrics on Bundles, Riemannian Manifolds, Reduction of Structure Groups, Orientation

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to vector bundles.

Definition 7.10 Given a (real) rank n vector bundle, $\xi = (E, \pi, B, V)$, we say that ξ is *Euclidean* iff there is a family, $(\langle -, - \rangle_b)_{b \in B}$, of inner products on each fibre, $\pi^{-1}(b)$, such that $\langle -, - \rangle_b$ depends smoothly on b, which means that for every trivializing map, $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$, for every frame, $(s_1, \ldots, s_n)$, on U_α, the maps

$$b \mapsto \langle s_i(b), s_j(b) \rangle_b, \qquad b \in U_\alpha,\ 1 \leq i, j \leq n$$

are smooth. We say that $\langle -, - \rangle$ is a *Euclidean metric* (or *Riemannian metric*) on ξ. If ξ is a complex rank n vector bundle, $\xi = (E, \pi, B, V)$, we say that ξ is *Hermitian* iff there is a family, $(\langle -, - \rangle_b)_{b \in B}$, of Hermitian inner products on each fibre, $\pi^{-1}(b)$, such that $\langle -, - \rangle_b$ depends smoothly on b. We say that $\langle -, - \rangle$ is a *Hermitian metric* on ξ. For any smooth manifold, M, if TM is a Euclidean vector bundle, then we say that M is a *Riemannian manifold*.

If M is a Riemannian manifold, the smoothness condition on the metric, $\{\langle -, -\rangle_p\}_{p\in M}$, on TM, can be expressed a little more conveniently. If $\dim(M) = n$, then for every chart, (U, φ), since $d\varphi^{-1}_{\varphi(p)} \colon \mathbb{R}^n \to T_pM$ is a bijection for every $p \in U$, the n-tuple of vector fields, $(s_1, \ldots, s_n)$, with $s_i(p) = d\varphi^{-1}_{\varphi(p)}(e_i)$, is a frame of TM over U, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Since every vector field over U is a linear combination, $\sum_{i=1}^n f_i s_i$, for some smooth functions, $f_i \colon U \to \mathbb{R}$, the condition of Definition 7.10 is equivalent to the fact that the maps,

$$p \mapsto \langle d\varphi^{-1}_{\varphi(p)}(e_i), d\varphi^{-1}_{\varphi(p)}(e_j)\rangle_p, \qquad p \in U,\ 1 \leq i, j \leq n,$$

are smooth. If we let $x = \varphi(p)$, the above condition says that the maps,

$$x \mapsto \langle d\varphi^{-1}_x(e_i), d\varphi^{-1}_x(e_j)\rangle_{\varphi^{-1}(x)}, \qquad x \in \varphi(U),\ 1 \leq i, j \leq n,$$

are smooth.

If M is a Riemannian manifold, the metric on TM is often denoted $g = (g_p)_{p\in M}$. In a chart, (U, φ), using local coordinates, we often use the notation, $g = \sum_{ij} g_{ij} dx_i \otimes dx_j$, or simply, $g = \sum_{ij} g_{ij} dx_i dx_j$, where

$$g_{ij}(p) = \left\langle \left(\frac{\partial}{\partial x_i}\right)_p, \left(\frac{\partial}{\partial x_j}\right)_p \right\rangle_p.$$

For every $p \in U$, the matrix, $(g_{ij}(p))$, is symmetric, positive definite.

The standard Euclidean metric on $\mathbb{R}^n$, namely,

$$g = dx_1^2 + \cdots + dx_n^2,$$

makes $\mathbb{R}^n$ into a Riemannian manifold. Then, every submanifold, M, of $\mathbb{R}^n$ inherits a metric by restricting the Euclidean metric to M. For example, the sphere, S^{n-1}, inherits a metric that makes S^{n-1} into a Riemannian manifold. It is a good exercise to find the local expression of this metric for S^2 in polar coordinates.

A nontrivial example of a Riemannian manifold is the *Poincaré upper half-space*, namely, the set $H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric

$$g = \frac{dx^2 + dy^2}{y^2}.$$

A way to obtain a metric on a manifold, N, is to pull-back the metric, g, on another manifold, M, along a local diffeomorphism, $\varphi \colon N \to M$. Recall that φ is a local diffeomorphism iff

$$d\varphi_p \colon T_pN \to T_{\varphi(p)}M$$

is a bijective linear map for every $p \in N$. Given any metric g on M, if φ is a local diffeomorphism, we define the *pull-back metric*, $\varphi^* g$, on N induced by g as follows: For all $p \in N$, for all $u, v \in T_p N$,

$$(\varphi^* g)_p(u, v) = g_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)).$$

We need to check that $(\varphi^* g)_p$ is an inner product, which is very easy since $d\varphi_p$ is a linear isomorphism. Our map, φ, between the two Riemannian manifolds $(N, \varphi^* g)$ and (M, g) is a local isometry, as defined below.

Definition 7.11 Given two Riemannian manifolds, (M_1, g_1) and (M_2, g_2), a *local isometry* is a smooth map, $\varphi\colon M_1 \to M_2$, such that $d\varphi_p\colon T_p M_1 \to T_{\varphi(p)} M_2$ is an isometry between the Euclidean spaces $(T_p M_1, (g_1)_p)$ and $(T_{\varphi(p)} M_2, (g_2)_{\varphi(p)})$, for every $p \in M_1$, that is,

$$(g_1)_p(u, v) = (g_2)_{\varphi(p)}(d\varphi_p(u), d\varphi_p(v)),$$

for all $u, v \in T_p M_1$ or, equivalently, $\varphi^* g_2 = g_1$. Moreover, φ is an *isometry* iff it is a local isometry and a diffeomorphism.

The isometries of a Riemannian manifold, (M, g), form a group, $\mathrm{Isom}(M, g)$, called the *isometry group of* (M, g). An important theorem of Myers and Steenrod asserts that the isometry group, $\mathrm{Isom}(M, g)$, is a Lie group.

Given a map, $\varphi\colon M_1 \to M_2$, and a metric g_1 on M_1, in general, φ does not induce any metric on M_2. However, if φ has some extra properties, it does induce a metric on M_2. This is the case when M_2 arises from M_1 as a quotient induced by some group of isometries of M_1. For more on this, see Gallot, Hulin and Lafontaine [60], Chapter 2, Section 2.A.

Now, given a real (resp. complex) vector bundle, ξ, provided that B is a sufficiently nice topological space, namely that B is *paracompact* (see Section 3.6), a Euclidean metric (resp. Hermitian metric) exists on ξ. This is a consequence of the existence of partitions of unity (see Theorem 3.26).

Theorem 7.9 *Every real (resp. complex) vector bundle admits a Euclidean (resp. Hermitian) metric. In particular, every smooth manifold admits a Riemannian metric.*

Proof. Let (U_α) be a trivializing open cover for ξ and pick any frame, $(s_1^\alpha, \ldots, s_n^\alpha)$, over U_α. For every $b \in U_\alpha$, the basis, $(s_1^\alpha(b), \ldots, s_n^\alpha(b))$ defines a Euclidean (resp. Hermitian) inner product, $\langle -, - \rangle_b$, on the fibre $\pi^{-1}(b)$, by declaring $(s_1^\alpha(b), \ldots, s_n^\alpha(b))$ orthonormal w.r.t. this inner product. (For $x = \sum_{i=1}^n x_i s_i^\alpha(b)$ and $y = \sum_{i=1}^n y_i s_i^\alpha(b)$, let $\langle x, y \rangle_b = \sum_{i=1}^n x_i y_i$, resp. $\langle x, y \rangle_b = \sum_{i=1}^n x_i \overline{y}_i$, in the complex case.) The $\langle -, - \rangle_b$ (with $b \in U_\alpha$) define a metric on $\pi^{-1}(U_\alpha)$, denote it $\langle -, - \rangle_\alpha$. Now, using Theorem 3.26, glue these inner products using a partition of unity, (f_α), subordinate to (U_α), by setting

$$\langle x, y \rangle = \sum_\alpha f_\alpha \langle x, y \rangle_\alpha.$$

We verify immediately that $\langle -, - \rangle$ is a Euclidean (resp. Hermitian) metric on ξ. $\square$

The existence of metrics on vector bundles allows the so-called reduction of structure group. Recall that the transition maps of a real (resp. complex) vector bundle, ξ, are functions, $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$ (resp. $\mathrm{GL}(n, \mathbb{C})$). Let $\mathrm{GL}^+(n, \mathbb{R})$ be the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of those matrices of positive determinant (resp. $\mathrm{GL}^+(n, \mathbb{C})$ be the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of those matrices of positive determinant).

Definition 7.12 For every real (resp. complex) vector bundle, ξ, if it is possible to find a cocycle, $g = (g_{\alpha\beta})$, for ξ with values in a subgroup, H, of $\mathrm{GL}(n, \mathbb{R})$ (resp. of $\mathrm{GL}(n, \mathbb{C})$), then we say that the *structure group of ξ can be reduced to H*. We say that ξ is *orientable* if its structure group can be reduced to $\mathrm{GL}^+(n, \mathbb{R})$ (resp. $\mathrm{GL}^+(n, \mathbb{C})$).

Proposition 7.10 *(a) The structure group of a rank n real vector bundle, ξ, can be reduced to $\mathbf{O}(n)$; it can be reduced to $\mathbf{SO}(n)$ iff ξ is orientable.*

(b) The structure group of a rank n complex vector bundle, ξ, can be reduced to $\mathbf{U}(n)$; it can be reduced to $\mathbf{SU}(n)$ iff ξ is orientable.

Proof. We prove (a), the proof of (b) being similar. Using Theorem 7.9, put a metric on ξ. For every U_α in a trivializing cover for ξ and every $b \in B$, by Gram-Schmidt, orthonormal bases for $\pi^{-1}(b)$ exit. Consider the family of trivializing maps, $\widetilde{\varphi}_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$, such that $\widetilde{\varphi}_{\alpha,b}\colon \pi^{-1}(b) \longrightarrow V$ maps orthonormal bases of the fibre to orthonormal bases of V. Then, it is easy to check that the corresponding cocycle takes values in $\mathbf{O}(n)$ and if ξ is orientable, the determinants being positive, these values are actually in $\mathbf{SO}(n)$. $\square$

Remark: If ξ is a Euclidean rank n vector bundle, then by Proposition 7.10, we may assume that ξ is given by some cocycle, $(g_{\alpha\beta})$, where $g_{\alpha\beta}(b) \in \mathbf{O}(n)$, for all $b \in U_\alpha \cap U_\beta$. We saw in Section 7.3 (f) that the dual bundle, ξ^*, is given by the cocycle

$$(g_{\alpha\beta}(b)^\top)^{-1}, \qquad b \in U_\alpha \cap U_\beta.$$

As $g_{\alpha\beta}(b)$ is an orthogonal matrix, $(g_{\alpha\beta}(b)^\top)^{-1} = g_{\alpha\beta}(b)$, and thus, any Euclidean bundle is isomorphic to its dual. As we noted earlier, this is *false* for Hermitian bundles.

Let $\xi = (E, \pi, B, V)$ be a rank n vector bundle and assume ξ is orientable. A family of trivializing maps, $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$, is *oriented* iff for all α, β, the transition function, $g_{\alpha\beta}(b)$ has positive determinant for all $b \in U_\alpha \cap U_\beta$. Two oriented families of trivializing maps, $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$ and $\psi_\beta\colon \pi^{-1}(W_\beta) \to W_\alpha \times V$, are *equivalent* iff for every $b \in U_\alpha \cap W_\beta$, the map $pr_2 \circ \varphi_\alpha \circ \psi_\beta^{-1} \upharpoonright \{b\} \times V\colon V \longrightarrow V$ has positive determinant. It is easily checked that this is an equivalence relation and that it partitions all the oriented families of trivializations of ξ into two equivalence classes. Either equivalence class is called an *orientation* of ξ.

If M is a manifold and $\xi = TM$, the tangent bundle of ξ, we know from Section 7.2 that the transition functions of TM are of the form

$$g_{\alpha\beta}(p)(u) = (\varphi_\alpha \circ \varphi_\beta^{-1})'_{\varphi(p)}(u),$$

where each $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^n$ is a chart of M. Consequently, TM is orientable iff the Jacobian of $(\varphi_\alpha \circ \varphi_\beta^{-1})'_{\varphi(p)}$ is positive, for every $p \in M$. This is equivalent to the condition of Definition 3.27 for M to be orientable. Therefore, the tangent bundle, TM, of a manifold, M, is orientable iff M is orientable.

The notion of orientability of a vector bundle, $\xi = (E, \pi, B, V)$, is *not* equivalent to the orientability of its total space, E. Indeed, if we look at the transition functions of the total space of TM given in Section 7.2, we see that TM, *as a manifold*, is always orientable, even if M is not orientable. Yet, *as a bundle*, TM is orientable iff M.

On the positive side, if $\xi = (E, \pi, B, V)$ is an orientable vector bundle and its base, B, is an orientable manifold, then E is orientable too.

To see this, assume that B is a manifold of dimension m, ξ is a rank n vector bundle with fibre V, let $((U_\alpha, \psi_\alpha))_\alpha$ be an atlas for B, let $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times V$ be a collection of trivializing maps for ξ and pick any isomorphism, $\iota\colon V \to \mathbb{R}^n$. Then, we get maps,

$$(\psi_\alpha \times \iota) \circ \varphi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow \mathbb{R}^m \times \mathbb{R}^n.$$

It is clear that these maps form an atlas for E. Check that the corresponding transition maps for E are of the form

$$(x, y) \mapsto (\psi_\beta \circ \psi_\alpha^{-1}(x), g_{\alpha\beta}(\psi_\alpha^{-1}(x))y).$$

Moreover, if B and ξ are orientable, check that these transition maps have positive Jacobian.

The fact that every bundle admits a metric allows us to define the notion of *orthogonal complement* of a subbundle. We state the following theorem without proof. The reader is invited to consult Milnor and Stasheff [110] for a proof (Chapter 3).

Proposition 7.11 *Let ξ and η be two vector bundles with ξ a subbundle of η. Then, there exists a subbundle, $\xi^\perp$, of η, such that every fibre of $\xi^\perp$ is the orthogonal complement of the fibre of ξ in the fibre of η, over every $b \in B$ and*

$$\eta \approx \xi \oplus \xi^\perp.$$

In particular, if N is a submanifold of a Riemannian manifold, M, then the orthogonal complement of TN in $TM \restriction N$ is isomorphic to the normal bundle, $(TM \restriction N)/TN$.

Remark: It can be shown (see Madsen and Tornehave [100], Chapter 15) that for every real vector bundle, ξ, there is some integer, k, such that ξ has a complement, η, in ϵ^k, where $\epsilon^k = B \times \mathbb{R}^k$ is the trivial rank k vector bundle, so that

$$\xi \oplus \eta = \epsilon^k.$$

This fact can be used to prove an interesting property of the space of global sections, $\Gamma(\xi)$. First, observe that $\Gamma(\xi)$ is not just a real vector space but also a $C^\infty(B)$-module (see Section 22.19). Indeed, for every smooth function, $f\colon B \to \mathbb{R}$, and every smooth section, $s\colon B \to E$, the map, $fs\colon B \to E$, given by

$$(fs)(b) = f(b)s(b), \qquad b \in B,$$

is a smooth section of ξ. In general, $\Gamma(\xi)$ is not a free $C^\infty(B)$-module unless ξ is trivial. However, the above remark implies that

$$\Gamma(\xi) \oplus \Gamma(\eta) = \Gamma(\epsilon^k),$$

where $\Gamma(\epsilon^k)$ is a free $C^\infty(B)$-module of dimension $\dim(\xi) + \dim(\eta)$. This proves that $\Gamma(\xi)$ is a finitely generated $C^\infty(B)$-module which is a summand of a free $C^\infty(B)$-module. Such modules are *projective modules*, see Definition 22.9 in Section 22.19. Therefore, $\Gamma(\xi)$ is a finitely generated projective $C^\infty(B)$-module. The following isomorphisms can be shown (see Madsen and Tornehave [100], Chapter 16):

Proposition 7.12 *The following isomorphisms hold for vector bundles:*

$$\begin{aligned}
\Gamma(\mathcal{H}om(\xi,\eta)) &\cong \mathrm{Hom}_{C^\infty(B)}(\Gamma(\xi),\Gamma(\eta)) \\
\Gamma(\xi\otimes\eta) &\cong \Gamma(\xi)\otimes_{C^\infty(B)}\Gamma(\eta) \\
\Gamma(\xi^*) &\cong \mathrm{Hom}_{C^\infty(B)}(\Gamma(\xi), C^\infty(B)) = (\Gamma(\xi))^* \\
\Gamma\Big(\bigwedge^k \xi\Big) &\cong \bigwedge^k_{C^\infty(B)}(\Gamma(\xi)).
\end{aligned}$$

7.5 Principal Fibre Bundles

We now consider principal bundles. Such bundles arise in terms of Lie groups acting on manifolds.

Definition 7.13 Let G be a Lie group. A *principal fibre bundle*, for short, a *principal bundle*, is a fibre bundle, $\xi = (E, \pi, B, G, G)$, in which the fibre is G and the structure group is also G, viewed as its group of left translations (ie., G acts on itself by multiplication on the left). This means that every transition function, $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, satisfies

$$g_{\alpha\beta}(b)(h) = g(b)h, \qquad \text{for some } g(b) \in G,$$

for all $b \in U_\alpha \cap U_\beta$ and all $h \in G$. A principal G-bundle is denoted $\xi = (E, \pi, B, G)$.

Note that G in $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ is viewed as its group of left translations under the isomorphism, $g \mapsto L_g$, and so, $g_{\alpha\beta}(b)$ is some left translation, $L_g(b)$. The inverse of the above isomorphism is given by $L \mapsto L(1)$, so $g(b) = g_{\alpha\beta}(b)(1)$. In view of these isomorphisms, we allow ourself the (convenient) abuse of notation

$$g_{\alpha\beta}(b)(h) = g_{\alpha\beta}(b)h,$$

where, on the left, $g_{\alpha\beta}(b)$ is viewed as a left translation of G and on the right, as an element of G.

When we want to emphasize that a principal bundle has structure group, G, we use the locution *principal G-bundle*.

It turns out that if $\xi = (E, \pi, B, G)$ is a principal bundle, then G acts on the total space, E, on the right. For the next proposition, recall that a right action, $\cdot \colon X \times G \to X$, is *free* iff for every $g \in G$, if $g \neq 1$, then $x \cdot g \neq x$ for all $x \in X$.

Proposition 7.13 *If $\xi = (E, \pi, B, G)$ is a principal bundle, then there is a right action of G on E. This action takes each fibre to itself and is free. Moreover, E/G is diffeomorphic to B.*

Proof. We show how to define the right action and leave the rest as an exercise. Let $\{(U_\alpha, \varphi_\alpha)\}$ be some trivializing cover defining ξ. For every $z \in E$, pick some U_α so that $\pi(z) \in U_\alpha$ and let $\varphi_\alpha(z) = (b, h)$, where $b = \pi(z)$ and $h \in G$. For any $g \in G$, we set

$$z \cdot g = \varphi_\alpha^{-1}(b, hg).$$

If we can show that this action does not depend on the choice of U_α, then it is clear that it is a free action. Suppose that we also have $b = \pi(z) \in U_\beta$ and that $\varphi_\beta(z) = (b, h')$. By definition of the transition functions, we have

$$h' = g_{\beta\alpha}(b)h \quad \text{and} \quad \varphi_\beta(z \cdot g) = (b, g_{\beta\alpha}(b)(hg)).$$

However,

$$g_{\beta\alpha}(b)(hg) = (g_{\beta\alpha}(b)h)g = h'g,$$

hence

$$z \cdot g = \varphi_\beta^{-1}(b, h'g),$$

which proves that our action does not depend on the choice of U_α. $\square$

Observe that the action of Proposition 7.13 is defined by

$$z \cdot g = \varphi_\alpha^{-1}(b, \varphi_{\alpha,b}(z)g), \quad \text{with} \quad b = \pi(z),$$

for all $z \in E$ and all $g \in G$. It is clear that this action satisfies the following two properties: For every $(U_\alpha, \varphi_\alpha)$,

(1) $\pi(z \cdot g) = \pi(z)$ and

(2) $\varphi_\alpha(z \cdot g) = \varphi_\alpha(z) \cdot g$, for all $z \in E$ and all $g \in G$,

where we define the right action of G on $U_\alpha \times G$ so that $(b, h) \cdot g = (b, hg)$. We say that φ_α is *G-equivariant* (or *equivariant*).

The following proposition shows that it is possible to define a principal G-bundle using a suitable right action and equivariant trivializations:

Proposition 7.14 *Let E be a smooth manifold, G a Lie group and let $\cdot\colon E \times G \to E$ be a smooth right action of G on E and assume that*

(a) The right action of G on E is free;

(b) The orbit space, $B = E/G$, is a smooth manifold under the quotient topology and the projection, $\pi\colon E \to E/G$, is smooth;

(c) There is a family of local trivializations, $\{(U_\alpha, \varphi_\alpha)\}$, where $\{U_\alpha\}$ is an open cover for $B = E/G$ and each
$$\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$$
is an equivariant diffeomorphism, which means that
$$\varphi_\alpha(z \cdot g) = \varphi_\alpha(z) \cdot g,$$
for all $z \in \pi^{-1}(U_\alpha)$ and all $g \in G$, where the right action of G on $U_\alpha \times G$ is $(b, h) \cdot g = (b, hg)$.

Then, $\xi = (E, \pi, E/G, G)$ is a principal G-bundle.

Proof. Since the action of G on E is free, every orbit, $b = z \cdot G$, is isomorphic to G and so, every fibre, $\pi^{-1}(b)$, is isomorphic to G. Thus, given that we have trivializing maps, we just have to prove that G acts by left translation on itself. Pick any (b, h) in $U_\beta \times G$ and let $z \in \pi^{-1}(U_\beta)$ be the unique element such that $\varphi_\beta(z) = (b, h)$. Then, as
$$\varphi_\beta(z \cdot g) = \varphi_\beta(z) \cdot g, \quad \text{for all } g \in G,$$
we have
$$\varphi_\beta(\varphi_\beta^{-1}(b, h) \cdot g) = \varphi_\beta(z \cdot g) = \varphi_\beta(z) \cdot g = (b, h) \cdot g,$$
which implies that
$$\varphi_\beta^{-1}(b, h) \cdot g = \varphi_\beta^{-1}((b, h) \cdot g).$$
Consequently,
$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, h) = \varphi_\alpha \circ \varphi_\beta^{-1}((b, 1) \cdot h) = \varphi_\alpha(\varphi_\beta^{-1}(b, 1) \cdot h) = \varphi_\alpha \circ \varphi_\beta^{-1}(b, 1) \cdot h,$$
and since
$$\varphi_\alpha \circ \varphi_\beta^{-1}(b, h) = (b, g_{\alpha\beta}(b)(h)) \quad \text{and} \quad \varphi_\alpha \circ \varphi_\beta^{-1}(b, 1) = (b, g_{\alpha\beta}(b)(1))$$

we get

$$g_{\alpha\beta}(b)(h) = g_{\alpha\beta}(b)(1)h.$$

The above shows that $g_{\alpha\beta}(b)\colon G \to G$ is the left translation by $g_{\alpha\beta}(b)(1)$ and thus, the transition functions, $g_{\alpha\beta}(b)$, constitute the group of left translations of G and ξ is indeed a principal G-bundle. $\square$

Bröcker and tom Dieck [25] (Chapter I, Section 4) and Duistermaat and Kolk [53] (Appendix A) define principal bundles using the conditions of Proposition 7.14. Propositions 7.13 and 7.14 show that this alternate definition is equivalent to ours (Definition 7.13).

It turns out that when we use the definition of a principal bundle in terms of the conditions of Proposition 7.14, it is convenient to define bundle maps in terms of equivariant maps. As we will see shortly, a map of principal bundles is a fibre bundle map.

Definition 7.14 If $\xi_1 = (E_1, \pi_1, B_1, G)$ and $\xi_2 = (E_2, \pi_2, B_1, G)$ are two principal bundles a *bundle map (or bundle morphism)*, $f\colon \xi_1 \to \xi_2$, is a pair, $f = (f_E, f_B)$, of smooth maps $f_E\colon E_1 \to E_2$ and $f_B\colon B_1 \to B_2$ such that

(a) The following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f_B"'] & B_2 \end{tikzcd}$$

(b) The map, f_E, is *G-equivariant*, that is,

$$f_E(a \cdot g) = f_E(a) \cdot g, \qquad \text{for all } a \in E_1 \text{ and all } g \in G.$$

A bundle map is an *isomorphism* if it has an inverse as in Definition 7.2. If the bundles ξ_1 and ξ_2 are over the same base, B, then we also require $f_B = \mathrm{id}$.

At first glance, it is not obvious that a map of principal bundles satisfies condition (b) of Definition 7.3. If we define $\widetilde{f}_\alpha\colon U_\alpha \times G \to V_\beta \times G$ by

$$\widetilde{f}_\alpha = \varphi'_\beta \circ f_E \circ \varphi_\alpha^{-1},$$

then locally, f_E is expressed as

$$f_E = {\varphi'_\beta}^{-1} \circ \widetilde{f}_\alpha \circ \varphi_\alpha.$$

Furthermore, it is trivial that if a map is equivariant and invertible then its inverse is equivariant. Consequently, since

$$\widetilde{f}_\alpha = \varphi'_\beta \circ f_E \circ \varphi_\alpha^{-1},$$

as φ_α^{-1}, φ'_β and f_E are equivariant, $\widetilde{f}_\alpha$ is also equivariant and so, $\widetilde{f}_\alpha$ is a map of (trivial) principal bundles. Thus, it it enough to prove that for every map of principal bundles,

$$\varphi\colon U_\alpha \times G \to V_\beta \times G,$$

there is some smooth map, $\rho_\alpha\colon U_\alpha \to G$, so that

$$\varphi(b, g) = (f_B(b), \rho_\alpha(b)(g)), \qquad \text{for all } b \in U_\alpha \text{ and all } g \in G.$$

Indeed, we have the following

Proposition 7.15 *For every map of trivial principal bundles,*

$$\varphi\colon U_\alpha \times G \to V_\beta \times G,$$

there are smooth maps, $f_B\colon U_\alpha \to V_\beta$ and $r_\alpha\colon U_\alpha \to G$, so that

$$\varphi(b, g) = (f_B(b), r_\alpha(b)g), \qquad \textit{for all } b \in U_\alpha \textit{ and all } g \in G.$$

In particular, φ is a diffeomorphism on fibres.

Proof. As φ is a map of principal bundles,

$$\varphi(b, 1) = (f_B(b), r_\alpha(b)), \qquad \text{for all } b \in U_\alpha$$

for some smooth maps, $f_B\colon U_\alpha \to V_\beta$ and $r_\alpha\colon U_\alpha \to G$. Now, using equivariance, we get

$$\varphi(b, g) = \varphi((b, 1)g) = \varphi(g, 1) \cdot g = (f_B(b), r_\alpha(b)) \cdot g = (f_B(b), r_\alpha(b)g),$$

as claimed. $\square$

Consequently, the map, $\rho_\alpha\colon U_\alpha \to G$, given by

$$\rho_\alpha(b)(g) = r_\alpha(b)g \qquad \text{for all } b \in U_\alpha \text{ and all } g \in G$$

satisfies

$$\varphi(b, g) = (f_B(b), \rho_\alpha(b)(g)), \qquad \text{for all } b \in U_\alpha \text{ and all } g \in G$$

and a map of principal bundles is indeed a fibre bundle map (as in Definition 7.3). Since a principal bundle map is a fibre bundle map, Proposition 7.3 also yields the useful fact:

Proposition 7.16 *Any map, $f\colon \xi_1 \to \xi_2$, between two principal bundles over the same base, B, is an isomorphism.*

Even though we are not aware of any practical applications in computer vision, robotics, or medical imaging, we wish to digress briefly on the issue of the triviality of bundles and the existence of sections.

A natural question is to ask whether a fibre bundle, ξ, is isomorphic to a trivial bundle. If so, we say that ξ is trivial. (By the way, the triviality of bundles comes up in physics, in particular, field theory.) Generally, this is a very difficult question, but a first step can be made by showing that it reduces to the question of triviality for principal bundles.

Indeed, if $\xi = (E, \pi, B, F, G)$ is a fibre bundle with fibre, F, using Theorem 7.4, we can construct a principal fibre bundle, $P(\xi)$, using the transition functions, $\{g_{\alpha\beta}\}$, of ξ, but using G itself as the fibre (acting on itself by left translation) instead of F. We obtain the *principal bundle, $P(\xi)$, associated to ξ*. For example, the principal bundle associated with a vector bundle is the *frame bundle*, discussed at the end of Section 7.3. Then, given two fibre bundles ξ and ξ', we see that ξ and ξ' are isomorphic iff $P(\xi)$ and $P(\xi')$ are isomorphic (Steenrod [141], Part I, Section 8, Theorem 8.2). More is true: The fibre bundle ξ is trivial iff the principal fibre bundle $P(\xi)$ is trivial (this is easy to prove, do it! Otherwise, see Steenrod [141], Part I, Section 8, Corollary 8.4). Morever, there is a test for the triviality of a principal bundle, the existence of a (global) section.

The following proposition, although easy to prove, is crucial:

Proposition 7.17 *If ξ is a principal bundle, then ξ is trivial iff it possesses some global section.*

Proof. If $f\colon B \times G \to \xi$ is an isomorphism of principal bundles over the same base, B, then for every $g \in G$, the map $b \mapsto f(b, g)$ is a section of ξ.

Conversely, let $s\colon B \to E$ be a section of ξ. Then, observe that the map, $f\colon B \times G \to \xi$, given by

$$f(b, g) = s(b)g$$

is a map of principal bundles. By Proposition 7.16, it is an isomorphism, so ξ is trivial. $\square$

Generally, in geometry, many objects of interest arise as global sections of some suitable bundle (or sheaf): vector fields, differential forms, tensor fields, *etc.*

Given a principal bundle, $\xi = (E, \pi, B, G)$, and given a manifold, F, if G acts effectively on F from the left, again, using Theorem 7.4, we can construct a fibre bundle, $\xi[F]$, from ξ, with F as typical fibre and such that $\xi[F]$ has the same transitions functions as ξ. In the case of a principal bundle, there is another slightly more direct construction that takes us from principal bundles to fibre bundles (see Duistermaat and Kolk [53], Chapter 2, and Davis and Kirk [39], Chapter 4, Definition 4.6, where it is called the *Borel construction*). This construction is of independent interest so we describe it briefly (for an application of this construction, see Duistermaat and Kolk [53], Chapter 2).

As ξ is a principal bundle, recall that G acts on E from the right, so we have a right action of G on $E \times F$, *via*

$$(z, f) \cdot g = (z \cdot g, g^{-1} \cdot f).$$

Consequently, we obtain the orbit set, $E \times F/\sim$, denoted $E \times_G F$, where $\sim$ is the equivalence relation

$$(z, f) \sim (z', f') \quad \text{iff} \quad (\exists g \in G)(z' = z \cdot g,\ f' = g^{-1} \cdot f).$$

Note that the composed map,

$$E \times F \xrightarrow{pr_1} E \xrightarrow{\pi} B,$$

factors through $E \times_G F$, since

$$\pi(pr_1(z, f)) = \pi(z) = \pi(z \cdot g) = \pi(pr_1(z \cdot g, g^{-1} \cdot f)).$$

Let $p\colon E \times_G F \to B$ be the corresponding map. The following proposition is not hard to show:

Proposition 7.18 *If $\xi = (E, \pi, B, G)$ is a principal bundle and F is any manifold such that G acts effectively on F from the left, then, $\xi[F] = (E \times_G F, p, B, F, G)$ is a fibre bundle with fibre F and structure group G and $\xi[F]$ and ξ have the same transition functions.*

Let us verify that the charts of ξ yield charts for $\xi[F]$. For any U_α in an open cover for B, we have a diffeomorphism

$$\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G.$$

Observe that we have an isomorphism

$$(U_\alpha \times G) \times_G F \cong U_\alpha \times F,$$

where, as usual, G acts on $U_\alpha \times G$ *via* $(z, h) \cdot g = (z, hg)$, an isomorphism

$$p^{-1}(U_\alpha) \cong \pi^{-1}(U_\alpha) \times_G F,$$

and that φ_α induces an isomorphism

$$\pi^{-1}(U_\alpha) \times_G F \xrightarrow{\varphi_\alpha} (U_\alpha \times G) \times_G F.$$

So, we get the commutative diagram

$$\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[r, "\sim"] \arrow[d, "p"'] & U_\alpha \times F \arrow[d, "pr_1"] \\
U_\alpha \arrow[r, equal] & U_\alpha,
\end{tikzcd}$$

which yields a local trivialization for $\xi[F]$. It is easy to see that the transition functions of $\xi[F]$ are the same as the transition functions of ξ.

The fibre bundle, $\xi[F]$, is called the fibre bundle *induced by* ξ. Now, if we start with a fibre bundle, ξ, with fibre, F, and structure group, G, if we make the associated principal bundle, $P(\xi)$, and then the induced fibre bundle, $P(\xi)[F]$, what is the relationship between ξ and $P(\xi)[F]$?

The answer is: ξ and $P(\xi)[F]$ are *equivalent* (this is because the transition functions are the same.)

Now, if we start with a principal G-bundle, ξ, make the fibre bundle, $\xi[F]$, as above, and then the principal bundle, $P(\xi[F])$, we get a principal bundle equivalent to ξ. Therefore, the maps

$$\xi \mapsto \xi[F] \quad \text{and} \quad \xi \mapsto P(\xi),$$

are mutual inverses and they set up a bijection between equivalence classes of principal G-bundles over B and equivalence classes of fibre bundles over B (with structure group, G). Moreover, this map extends to morphisms, so it is functorial (see Steenrod [141], Part I, Section 2, Lemma 2.6–Lemma 2.10). As a consequence, in order to "classify" equivalence classes of fibre bundles (assuming B and G fixed), it is enough to know how to classify principal G-bundles over B. Given some reasonable conditions on the coverings of B, Milnor solved this classification problem, but this is taking us way beyond the scope of these notes!

The classical reference on fibre bundles, vector bundles and principal bundles, is Steenrod [141]. More recent references include Bott and Tu [19], Madsen and Tornehave [100], Morita [114], Griffith and Harris [66], Wells [150], Hirzebruch [77], Milnor and Stasheff [110], Davis and Kirk [39], Atiyah [10], Chern [33], Choquet-Bruhat, DeWitt-Morette and Dillard-Bleick [37], Hirsh [76], Sato [133], Narasimham [117], Sharpe [139] and also Husemoller [82], which covers more, including characteristic classes.

Proposition 7.14 shows that principal bundles are induced by suitable right actions but we still need sufficient conditions to guarantee conditions (a), (b) and (c). Such conditions are given in the next section.

7.6 Homogeneous Spaces, II

Now that we know about manifolds and Lie groups, we can revisit the notion of homogeneous space given in Definition 2.8, which only applied to groups and sets without any topology or differentiable structure.

Definition 7.15 A *homogeneous space* is a smooth manifold, M, together with a smooth transitive action, $\cdot\colon G \times M \to M$, of a Lie group, G, on M.

In this section, we prove that G is the total space of a principal bundle with base space M and structure group, G_x, the stabilizer of any $x \in M$.

If M is a manifold, G is a Lie group and $\cdot\colon M \times G \to M$ is a smooth right action, in general, M/G is not even Hausdorff. A sufficient condition can be given using the notion of a proper map. If X and Y are two Hausdorff topological spaces,[1] a continuous map, $\varphi\colon X \to Y$, is *proper* iff for every topological space, Z, the map $\varphi \times \mathrm{id}\colon X \times Z \to Y \times Z$ is a *closed map* (A map, f, is a closed map iff the image of any closed set by f is a closed set). If we let Z be a one-point space, we see that a proper map is closed. It can be shown (see Bourbaki, General Topology [23], Chapter 1, Section 10) that a continuous map, $\varphi\colon X \to Y$, is proper iff φ is closed and if $\varphi^{-1}(y)$ is compact for every $y \in Y$. If φ is proper, it is easy to show that $\varphi^{-1}(K)$ is compact in X whenever K is compact in Y. Moreover, if Y is also locally compact, then Y is compactly generated, which means that a subset, C, of Y is closed iff $K \cap C$ is closed in C for every compact subset K of Y (see Munkres [115]). In this case (Y locally compact), φ is a closed map iff $\varphi^{-1}(K)$ is compact in X whenever K is compact in Y (see Bourbaki, General Topology [23], Chapter 1, Section 10).[2] In particular, this is true if Y is a manifold since manifolds are locally compact. Then, we say that the action, $\cdot\colon M \times G \to M$, is *proper* iff the map,

$$M \times G \longrightarrow M \times M, \quad (x, g) \mapsto (x, x \cdot g),$$

is proper.

If G and M are Hausdorff and G is locally compact, then it can be shown (see Bourbaki, General Topology [23], Chapter 3, Section 4) that the action $\cdot\colon M \times G \to M$ is proper iff for all $x, y \in M$, there exist some open sets, V_x and V_y in M, with $x \in V_x$ and $y \in V_y$, so that the closure, $\overline{K}$, of the set $K = \{g \in G \mid V_x \cdot g \cap V_y \neq \emptyset\}$ is compact in G. In particular, if G has the discrete topology, this conditions holds iff the sets $\{g \in G \mid V_x \cdot g \cap V_y \neq \emptyset\}$ are finite. Also, if G is compact, then $\overline{K}$ is automatically compact, so every compact group acts properly. If the action, $\cdot\colon M \times G \to M$, is proper, then the orbit equivalence relation is closed since it is the image of $M \times G$ in $M \times M$, and so, M/G is Hausdorff. We then have the following theorem proved in Duistermaat and Kolk [53] (Chapter 1, Section 11):

Theorem 7.19 *Let M be a smooth manifold, G be a Lie group and let $\cdot\colon M \times G \to M$ be a right smooth action which is proper and free. Then, M/G is a principal G-bundle of dimension* $\dim M - \dim G$.

Theorem 7.19 has some interesting corollaries. Let G be a Lie group and let H be a closed subgroup of G. Then, there is a right action of H on G, namely

$$G \times H \longrightarrow G, \quad (g, h) \mapsto gh,$$

and this action is clearly free and proper. Because a closed subgroup of a Lie group is a Lie group, we get the following result whose proof can be found in Bröcker and tom Dieck [25] (Chapter I, Section 4) or Duistermaat and Kolk [53] (Chapter 1, Section 11):

[1] It is not necessary to assume that X and Y are Hausdorff but, if X and/or Y are not Hausdorff, we have to replace "compact" by "quasi-compact." We have no need for this extra generality.

[2] Duistermaat and Kolk [53] seem to have overlooked the fact that a condition on Y (such as local compactness) is needed in their remark on lines 5-6, page 53, just before Lemma 1.11.3.

Corollary 7.20 *If G is a Lie group and H is a closed subgroup of G, then, the right action of H on G defines a principal H-bundle, $\xi = (G, \pi, G/H, H)$, where $\pi\colon G \to G/H$ is the canonical projection. Moreover, π is a submersion, which means that $d\pi_g$ is surjective for all $g \in G$ (equivalently, the rank of $d\pi_g$ is constant and equal to $\dim G/H$, for all $g \in G$).*

Now, if $\cdot\colon G \times M \to M$ is a smooth transitive action of a Lie group, G, on a manifold, M, we know that the stabilizers, G_x, are all isomorphic and closed (see Section 2.5, Remark after Theorem 2.26). Then, we can let $H = G_x$ and apply Corollary 7.20 to get the following result (mostly proved in in Bröcker and tom Dieck [25] (Chapter I, Section 4):

Proposition 7.21 *Let $\cdot\colon G \times M \to M$ be smooth transitive action of a Lie group, G, on a manifold, M. Then, G/G_x and M are diffeomorphic and G is the total space of a principal bundle, $\xi = (G, \pi, M, G_x)$, where G_x is the stabilizer of any element $x \in M$.*

Thus, we finally see that homogeneous spaces induce principal bundles. Going back to some of the examples of Section 2.2, we see that

(1) $\mathbf{SO}(n+1)$ is a principal $\mathbf{SO}(n)$-bundle over the sphere S^n (for $n \geq 0$).

(2) $\mathbf{SU}(n+1)$ is a principal $\mathbf{SU}(n)$-bundle over the sphere S^{2n+1} (for $n \geq 0$).

(3) $\mathbf{SL}(2, \mathbb{R})$ is a principal $\mathbf{SO}(2)$-bundle over the upper-half space, H.

(4) $\mathbf{GL}(n, \mathbb{R})$ is a principal $\mathbf{O}(n)$-bundle over the space $\mathbf{SPD}(n)$ of symmetric, positive definite matrices.

(5) $\mathbf{GL}^+(n, \mathbb{R})$, is a principal $\mathbf{SO}(n)$-bundle over the space, $\mathbf{SPD}(n)$, of symmetric, positive definite matrices, with fibre $\mathbf{SO}(n)$.

(6) $\mathbf{SO}(n+1)$ is a principal $\mathbf{O}(n)$-bundle over the real projective space $\mathbb{RP}^n$ (for $n \geq 0$).

(7) $\mathbf{SU}(n+1)$ is a principal $\mathbf{U}(n)$-bundle over the complex projective space $\mathbb{CP}^n$ (for $n \geq 0$).

(8) $\mathbf{O}(n)$ is a principal $\mathbf{O}(k) \times \mathbf{O}(n-k)$-bundle over the Grassmannian, $G(k, n)$.

(9) $\mathbf{SO}(n)$ is a principal $S(\mathbf{O}(k) \times \mathbf{O}(n-k))$-bundle over the Grassmannian, $G(k, n)$.

(10) From Section 2.5, we see that the Lorentz group, $\mathbf{SO}_0(n, 1)$, is a principal $\mathbf{SO}(n)$-bundle over the space, $\mathcal{H}_n^+(1)$, consisting of one sheet of the hyperbolic paraboloid $\mathcal{H}_n(1)$.

Thus, we see that both $\mathbf{SO}(n+1)$ and $\mathbf{SO}_0(n, 1)$ are principal $\mathbf{SO}(n)$-bundles, the difference being that the base space for $\mathbf{SO}(n+1)$ is the sphere, S^n, which is compact, whereas the base space for $\mathbf{SO}_0(n, 1)$ is the (connected) surface, $\mathcal{H}_n^+(1)$, which is not compact. Many more examples can be given, for instance, see Arvanitoyeogos [8].

Chapter 8

Differential Forms

8.1 Differential Forms on Subsets of $\mathbb{R}^n$ and de Rham Cohomology

The theory of differential forms is one of the main tools in geometry and topology. This theory has a surprisingly large range of applications and it also provides a relatively easy access to more advanced theories such as cohomology. For all these reasons, it is really an indispensable theory and anyone with more than a passible interest in geometry should be familiar with it.

The theory of differential forms was initiated by Poincaré and further elaborated by Elie Cartan at the end of the nineteenth century. Differential forms have two main roles:

(1) Describe various systems of partial differential equations on manifolds.

(2) To define various geometric invariants reflecting the global structure of manifolds or bundles. Such invariants are obtained by integrating certain differential forms.

As we will see shortly, as soon as one tries to define integration on higher-dimensional objects, such as manifolds, one realizes that it is not functions that are integrated but instead, differential forms. Furthermore, as by magic, the algebra of differential forms handles changes of variables automatically and yields a neat form of "Stokes formula".

Our goal is to define differential forms on manifolds but we begin with differential forms on open subsets of $\mathbb{R}^n$ in order to build up intuition.

Differential forms are smooth functions on open subset, U, of $\mathbb{R}^n$, taking as values alternating tensors in some exterior power, $\bigwedge^p(\mathbb{R}^n)^*$. Recall from Sections 22.14 and 22.15, in particular, Proposition 22.24, that for every finite-dimensional vector space, E, the isomorphisms, $\mu\colon \bigwedge^n(E^*) \longrightarrow \mathrm{Alt}^n(E;\mathbb{R})$, induced by the linear extensions of the maps given by

$$\mu(v_1^* \wedge \cdots \wedge v_n^*)(u_1, \ldots, u_n) = \det(u_j^*(u_i))$$

yield a canonical isomorphism of algebras, $\mu\colon \bigwedge(E^*) \longrightarrow \mathrm{Alt}(E)$, where

$$\mathrm{Alt}(E) = \bigoplus_{n\geq 0} \mathrm{Alt}^n(E;\mathbb{R})$$

and where $\mathrm{Alt}^n(E;\mathbb{R})$ is the vector space of alternating multilinear maps on $\mathbb{R}^n$. In view of these isomorphisms, we will identify ω and $\mu(\omega)$ for any $\omega \in \bigwedge^n(E^*)$ and we will write $\omega(u_1,\ldots,u_n)$ as an abbrevation for $\mu(\omega)(u_1,\ldots,u_n)$.

Because $\mathrm{Alt}(\mathbb{R}^n)$ is an algebra under the wedge product, differential forms also have a wedge product. However, the power of differential forms stems from the *exterior differential*, d, which is a skew-symmetric version of the usual differentiation operator.

Definition 8.1 Given any open subset, U, of $\mathbb{R}^n$, a smooth *differential p-form on U*, for short, *p-form on U*, is any smooth function, $\omega\colon U \to \bigwedge^p(\mathbb{R}^n)^*$. The vector space of all p-forms on U is denoted $\mathcal{A}^p(U)$. The vector space, $\mathcal{A}^*(U) = \bigoplus_{p\geq 0}\mathcal{A}^p(U)$, is the set of *differential forms on U*.

Observe that $\mathcal{A}^0(U) = C^\infty(U,\mathbb{R})$, the vector space of smooth functions on U and $\mathcal{A}^1(U) = C^\infty(U,(\mathbb{R}^n)^*)$, the set of smooth functions from U to the set of linear forms on $\mathbb{R}^n$. Also, $\mathcal{A}^p(U) = (0)$ for $p > n$.

Remark: The space, $\mathcal{A}^*(U)$, is also denoted $\mathcal{A}^\bullet(U)$. Other authors use $\Omega^p(U)$ instead of $\mathcal{A}^p(U)$ but we prefer to reserve Ω^p for holomorphic forms.

Recall from Section 22.12 that if $(e_1,\ldots,e_n)$ is any basis of $\mathbb{R}^n$ and $(e_1^*,\ldots,e_n^*)$ is its dual basis, then the alternating tensors,

$$e_I^* = e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*,$$

form basis of $\bigwedge^p(\mathbb{R}^n)^*$, where $I = \{i_1,\ldots,i_p\} \subseteq \{1,\ldots,n\}$, with $i_1 < \cdots < i_p$. Thus, with respect to the basis $(e_1,\ldots,e_n)$, every p-form, ω, can be uniquely written

$$\omega(x) = \sum_I f_I(x)\, e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* = \sum_I f_I(x)\, e_I^* \qquad x \in U,$$

where each f_I is a smooth function on U. For example, if $U = \mathbb{R}^2 - \{0\}$, then

$$\omega(x,y) = \frac{-y}{x^2+y^2}\, e_1^* + \frac{x}{x^2+y^2}\, e_2^*$$

is a 2-form on U, (with $e_1 = (1,0)$ and $e_2 = (0,1)$).

We often write ω_x instead of $\omega(x)$. Now, not only is $\mathcal{A}^*(U)$ a vector space, it is also an algebra.

Definition 8.2 The *wedge product* on $\mathcal{A}^*(U)$ is defined as follows: For all $p, q \geq 0$, the wedge product, $\wedge\colon \mathcal{A}^p(U) \times \mathcal{A}^q(U) \to \mathcal{A}^{p+q}(U)$, is given by

$$(\omega \wedge \eta)(x) = \omega(x) \wedge \eta(x), \qquad x \in U.$$

For example, if ω and η are one-forms, then

$$(\omega \wedge \eta)_x(u, v) = \omega_x(u) \wedge \eta_x(v) - \omega_x(v) \wedge \eta_x(u).$$

For $f \in \mathcal{A}^0(U) = C^\infty(U, \mathbb{R})$ and $\omega \in \mathcal{A}^p(U)$, we have $f \wedge \omega = f\omega$. Thus, the algebra, $\mathcal{A}^*(U)$, is also a $C^\infty(U, \mathbb{R})$-module,

Proposition 22.22 immediately yields

Proposition 8.1 *For all forms $\omega \in \mathcal{A}^p(U)$ and $\eta \in \mathcal{A}^q(U)$, we have*

$$\eta \wedge \omega = (-1)^{pq} \omega \wedge \eta.$$

We now come to the crucial operation of exterior differentiation. First, recall that if $f\colon U \to V$ is a smooth function from $U \subseteq \mathbb{R}^n$ to a (finite-dimensional) normed vector space, V, the derivative, $f'\colon U \to \mathrm{Hom}(\mathbb{R}^n, V)$, of f (also denoted, Df) is a function where $f'(x)$ is a linear map, $f'(x) \in \mathrm{Hom}(\mathbb{R}^n, V)$, for every $x \in U$, and such that

$$f'(x)(e_j) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(x)\, u_i, \qquad 1 \leq j \leq n,$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$ and $(u_1, \ldots, u_m)$ is a basis of V. The $m \times n$ matrix,

$$\left(\frac{\partial f_i}{\partial x_j}\right),$$

is the *Jacobian matrix* of f. We also write $f'_x(u)$ for $f'(x)(u)$. Observe that since a p-form is a smooth map, $\omega\colon U \to \bigwedge^p(\mathbb{R}^n)^*$, its derivative is a map,

$$\omega'\colon U \to \mathrm{Hom}(\mathbb{R}^n, \bigwedge^p(\mathbb{R}^n)^*),$$

such that ω'_x is a linear map from $\mathbb{R}^n$ to $\bigwedge^p(\mathbb{R}^n)^*$, for every $x \in U$. By the isomorphism, $\bigwedge^p(\mathbb{R}^n)^* \cong \mathrm{Alt}^p(\mathbb{R}^n; \mathbb{R})$, we can view ω'_x as a linear map, $\omega_x\colon \mathbb{R}^n \to \mathrm{Alt}^p(\mathbb{R}^n; \mathbb{R})$, or equivalently, as a multilinear form, $\omega'_x\colon (\mathbb{R}^n)^{p+1} \to \mathbb{R}$, which is alternating in its last p arguments. The exterior derivative, $(d\omega)_x$, is obtained by making ω'_x into an alternating map in all of its $p+1$ arguments.

Definition 8.3 For every $p \geq 0$, the *exterior differential*, $d\colon \mathcal{A}^p(U) \to \mathcal{A}^{p+1}(U)$, is given by

$$(d\omega)_x(u_1, \ldots, u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} \omega'_x(u_i)(u_1, \ldots, \widehat{u_i}, \ldots, u_{p+1}),$$

for all $\omega \in \mathcal{A}^p(U)$ and all $u_1, \ldots, u_{p+1} \in \mathbb{R}^n$, where the hat over the argument u_i means that it should be omitted.

One should check that $(d\omega)_x$ is indeed alternating but this is easy. If necessary to avoid confusion, we write $d^p\colon \mathcal{A}^p(U) \to \mathcal{A}^{p+1}(U)$ instead of $d\colon \mathcal{A}^p(U) \to \mathcal{A}^{p+1}(U)$.

Remark: Definition 8.3 is the definition adopted by Cartan [29, 30][1] and Madsen and Tornehave [100]. Some authors use a different approach often using Propositions 8.2 and 8.3 as a starting point but we find the approach using Definition 8.3 more direct. Furthermore, this approach extends immediately to the case of vector valued forms.

For any smooth function, $f \in \mathcal{A}^0(U) = C^\infty(U, \mathbb{R})$, we get

$$df_x(u) = f'_x(u).$$

Therefore, for smooth functions, *the exterior differential, df, coincides with the usual derivative, f'* (we identify $\bigwedge^1(\mathbb{R}^n)^*$ and $(\mathbb{R}^n)^*$). For any 1-form, $\omega \in \mathcal{A}^1(U)$, we have

$$d\omega_x(u, v) = \omega'_x(u)(v) - \omega'_x(v)(u).$$

It follows that the map

$$(u, v) \mapsto \omega'_x(u)(v)$$

is symmetric iff $d\omega = 0$.

For a concrete example of exterior differentiation, if

$$\omega(x, y) = \frac{-y}{x^2 + y^2}\, e_1^* + \frac{x}{x^2 + y^2}\, e_2^*,$$

check that $d\omega = 0$.

The following observation is quite trivial but it will simplify notation: On $\mathbb{R}^n$, we have the projection function, $pr_i\colon \mathbb{R}^n \to \mathbb{R}$, with $pr_i(u_1, \ldots, u_n) = u_i$. Note that $pr_i = e_i^*$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Let $x_i\colon U \to \mathbb{R}$ be the restriction of pr_i to U. Then, note that *x'_i is the constant map* given by

$$x'_i(x) = pr_i, \qquad x \in U.$$

[1] We warn the reader that a few typos have crept up in the English translation, Cartan [30], of the orginal version Cartan [29].

It follows that $dx_i = x_i'$ is the constant function with value $pr_i = e_i^*$. Now, since every p-form, ω, can be uniquely expressed as

$$\omega_x = \sum_I f_I(x)\, e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* = \sum_I f_I(x) e_I^*, \qquad x \in U,$$

using Definition 8.2, we see immediately that ω can be uniquely written in the form

$$\omega = \sum_I f_I(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \tag{$*$}$$

where the f_I are smooth functions on U.

Observe that for $f \in \mathcal{A}^0(U) = C^\infty(U, \mathbb{R})$, we have

$$df_x = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)\, e_i^* \quad \text{and} \quad df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i.$$

Proposition 8.2 *For every p form, $\omega \in \mathcal{A}^p(U)$, with $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, we have*

$$d\omega = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Proof. Recall that $\omega_x = f e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* = f e_I^*$, so

$$\omega_x'(u) = f_x'(u) e_I^* = df_x(u) e_I^*$$

and by Definition 8.3, we get

$$d\omega_x(u_1, \ldots, u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} df_x(u_i) e_I^*(u_1, \ldots, \widehat{u_i}, \ldots, u_{p+1}) = (df_x \wedge e_I^*)(u_1, \ldots, u_{p+1}),$$

where the last equation is an instance of the equation stated just before Proposition 22.24. $\square$

We can now prove

Proposition 8.3 *For all $\omega \in \mathcal{A}^p(U)$ and all $\eta \in \mathcal{A}^q(U)$,*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

Proof. In view of the unique representation, $(*)$, it is enough to prove the proposition when $\omega = f e_I^*$ and $\eta = g e_J^*$. In this case, as $\omega \wedge \eta = f g\, e_I^* \wedge e_J^*$, by Proposition 8.2, we have

$$\begin{aligned}
d(\omega \wedge \eta) &= d(fg) \wedge e_I^* \wedge e_J^* \\
&= ((df)g + f(dg)) \wedge e_I^* \wedge e_J^* \\
&= (df) g e_I^* \wedge e_J^* + f(dg) \wedge e_I^* \wedge e_J^* \\
&= (df) e_I^* \wedge g e_J^* + (-1)^p f \wedge e_I^* \wedge (dg) \wedge e_J^* \\
&= d\omega \wedge \eta + (-1)^p \omega \wedge d\eta,
\end{aligned}$$

as claimed. $\square$

We say that d is an *anti-derivation of degree* -1. Finally, here is the crucial and almost magical property of d:

Proposition 8.4 *For every $p \geq 0$, the composition $\mathcal{A}^p(U) \xrightarrow{d} \mathcal{A}^{p+1}(U) \xrightarrow{d} \mathcal{A}^{p+2}(U)$ is identically zero, that is,*

$$d \circ d = 0,$$

or, using superscripts, $d^{p+1} \circ d^p = 0$.

Proof. It is enough to prove the proposition when $\omega = f e_I^*$. We have

$$d\omega_x = df_x \wedge e_I^* = \frac{\partial f}{\partial x_1}(x)\, e_1^* \wedge e_I^* + \cdots + \frac{\partial f}{\partial x_n}(x)\, e_n^* \wedge e_I^*.$$

As $e_i^* \wedge e_j^* = -e_j^* \wedge e_i^*$ and $e_i^* \wedge e_i^* = 0$, we get

$$\begin{aligned} (d \circ d)\omega &= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\, e_i^* \wedge e_j^* \wedge e_I^* \\ &= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{\partial^2 f}{\partial x_j \partial x_i}(x) \right) e_i^* \wedge e_j^* \wedge e_I^* = 0, \end{aligned}$$

since partial derivatives commute (as f is smooth).

Propositions 8.2, 8.3 and 8.4 can be summarized by saying that $\mathcal{A}^*(U)$ together with the product, $\wedge$, and the differential, d, is a *differential graded algebra*. As $\mathcal{A}^*(U)) = \bigoplus_{p \geq 0} \mathcal{A}^p(U)$ and $d^p \colon \mathcal{A}^p(U) \to \mathcal{A}^{p+1}(U)$, we can view $d = (d^p)$ as a linear map, $d \colon \mathcal{A}^*(U) \to \mathcal{A}^*(U)$, such that

$$d \circ d = 0.$$

The diagram

$$\mathcal{A}^0(U) \xrightarrow{d} \mathcal{A}^1(U) \longrightarrow \cdots \longrightarrow \mathcal{A}^{p-1}(U) \xrightarrow{d} \mathcal{A}^p(U) \xrightarrow{d} \mathcal{A}^{p+1}(U) \longrightarrow \cdots$$

is called the *de Rham complex* of U. It is a *cochain complex*.

Let us consider one more example. Assume $n = 3$ and consider any function, $f \in \mathcal{A}^0(U)$. We have

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$$

and the vector

$$\left(\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z} \right)$$

is the *gradient* of f. Next, let

$$\omega = P dx + Q dy + R dz$$

be a 1-form on some open, $U \subseteq \mathbb{R}^3$. An easy calculation yields

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

The vector field given by

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

is the *curl* of the vector field given by (P, Q, R). Now, if

$$\eta = A dy \wedge dz + B dz \wedge dx + C dx \wedge dy$$

is a 2-form on $\mathbb{R}^3$, we get

$$d\eta = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz.$$

The real number,

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$$

is called the *divergence* of the vector field (A, B, C). When is there a smooth field, (P, Q, R), whose curl is given by a prescribed smooth field, (A, B, C)? Equivalently, when is there a 1-form, $\omega = P dx + Q dy + R dz$, such that

$$d\omega = \eta = A dy \wedge dz + B dz \wedge dx + C dx \wedge dy?$$

By Proposition 8.4, it is necessary that $d\eta = 0$, that is, that (A, B, C) has zero divergence. However, this condition is not sufficient in general; it depends on the topology of U. If U is *star-like*, Poincaré's Lemma (to be considered shortly) says that this condition is sufficient.

Definition 8.4 A differential form, ω, is *closed* iff $d\omega = 0$, *exact* iff $\omega = d\eta$, for some differential form, η. For every $p \geq 0$, let

$$Z^p(U) = \{\omega \in \mathcal{A}^p(U) \mid d\omega = 0\} = \mathrm{Ker}\, d\colon \mathcal{A}^p(U) \longrightarrow \mathcal{A}^{p+1}(U),$$

be the vector space of closed p-forms, also called *p-cocycles* and for every $p \geq 1$, let

$$B^p(U) = \{\omega \in \mathcal{A}^p(U) \mid \exists \eta \in \mathcal{A}^{p-1}(U),\ \omega = d\eta\} = \mathrm{Im}\, d\colon \mathcal{A}^{p-1}(U) \longrightarrow \mathcal{A}^p(U),$$

be the vector space of exact p-forms, also called *p-coboundaries*. Set $B^0(U) = (0)$. Forms in $\mathcal{A}^p(U)$ are also called *p-cochains*. As $B^p(U) \subseteq Z^p(U)$ (by Proposition 8.4), for every $p \geq 0$, we define the p^{th} *de Rham cohomology group* of U as the quotient space

$$H^p_{\mathrm{DR}}(U) = Z^p(U)/B^p(U).$$

An element of $H^p_{\mathrm{DR}}(U)$ is called a *cohomology class* and is denoted $[\omega]$, where $\omega \in Z^p(U)$ is a cocycle. The real vector space, $H^\bullet_{\mathrm{DR}}(U) = \bigoplus_{p \geq 0} H^p_{\mathrm{DR}}(U)$, is called the *de Rham cohomology algebra* of U.

We often drop the subscript DR and write $H^p(U)$ for $H^p_{\mathrm{DR}}(U)$ (resp. $H^\bullet(U)$ for $H^\bullet_{\mathrm{DR}}(U)$) when no confusion arises. Proposition 8.4 shows that every exact form is closed but the converse is false in general. Measuring the extent to which closed forms are not exact is the object of *de Rham cohomology*. For example, if we consider the form

$$\omega(x,y) = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy,$$

on $U = \mathbb{R}^2 - \{0\}$, we have $d\omega = 0$. Yet, it is not hard to show (using integration, see Madsen and Tornehave [100], Chapter 1) that there is no smooth function, f, on U such that $df = \omega$. Thus, ω is a closed form which is not exact. This is because U is punctured.

Observe that $H^0(U) = Z^0(U) = \{f \in C^\infty(U,\mathbb{R}) \mid df = 0\}$, that is, $H^0(U)$ is the space of locally constant functions on U, equivalently, the space of functions that are constant on the connected components of U. Thus, the cardinality of $H^0(U)$ gives the number of connected components of U. For a large class of open sets (for example, open sets that can be covered by finitely many convex sets), the cohomology groups, $H^p(U)$, are finite dimensional.

Going back to Definition 8.4, we define the vector spaces $Z^*(U)$ and $B^*(U)$ by

$$Z^*(U) = \bigoplus_{p\geq 0} Z^p(U) \quad \text{and} \quad B^*(U) = \bigoplus_{p\geq 0} B^p(U).$$

Now, $\mathcal{A}^*(U)$ is a graded algebra with multiplication, $\wedge$. Observe that $Z^*(U)$ is a subalgebra of $\mathcal{A}^*(U)$, since

$$d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^p\omega\wedge d\eta,$$

so $d\omega = 0$ and $d\eta = 0$ implies $d(\omega\wedge\eta) = 0$. Furthermore, $B^*(U)$ is an ideal in $Z^*(U)$, because if $\omega = d\eta$ and $d\tau = 0$, then

$$d(\eta\tau) = d\eta\wedge\tau + (-1)^{p-1}\eta\wedge d\tau = \omega\wedge\tau,$$

with $\eta \in \mathcal{A}^{p-1}(U)$. Therefore, $H^\bullet_{\mathrm{DR}} = Z^*(U)/B^*(U)$ inherits a graded algebra structure from $\mathcal{A}^*(U)$. Explicitly, the multiplication in $H^\bullet_{\mathrm{DR}}$ is given by

$$[\omega]\,[\eta] = [\omega\wedge\eta].$$

It turns out that Propositions 8.3 and 8.4 together with the fact that d coincides with the derivative on $\mathcal{A}^0(U)$ characterize the differential, d.

Theorem 8.5 *There is a unique linear map, $d\colon \mathcal{A}^*(U) \to \mathcal{A}^*(U)$, with $d = (d^p)$ and $d^p\colon \mathcal{A}^p(U) \to \mathcal{A}^{p+1}(U)$ for every $p \geq 0$, such that*

(1) $df = f'$, for every $f \in \mathcal{A}^0(U) = C^\infty(U,\mathbb{R})$.

(2) $d\circ d = 0$.

(3) For every $\omega \in \mathcal{A}^p(U)$ and every $\eta \in \mathcal{A}^q(U)$,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

Proof. Existence has already been shown so we only have to prove uniqueness. Let δ be another linear map satisfying (1)–(3). By (1), $df = \delta f = f'$, if $f \in \mathcal{A}^0(U)$. In particular, this hold when $f = x_i$, with $x_i \colon U \to \mathbb{R}$ the restriction of pr_i to U. In this case, we know that $\delta x_i = e_i^*$, the constant function, $e_i^* = pr_i$. By (2), $\delta e_i^* = 0$. Using (3), we get $\delta e_I^* = 0$, for every nonempty subset $I \subseteq \{1, \ldots, n\}$. If $\omega = f e_I^*$, by (3), we get

$$\delta\omega = \delta f \wedge e_I^* + f \wedge \delta e_I^* = \delta f \wedge e_I^* = df \wedge e_I^* = d\omega.$$

Finally, since every differential form is a linear combination of special forms, $f_I e_I^*$, we conclude that $\delta = d$. $\square$

We now consider the action of smooth maps, $\varphi \colon U \to U'$, on differential forms in $\mathcal{A}^*(U')$. We will see that φ induces a map from $\mathcal{A}^*(U')$ to $\mathcal{A}^*(U)$ called a *pull-back map*. This correspond to a change of variables.

Recall Proposition 22.21 which states that if $f \colon E \to F$ is any linear map between two finite-dimensional vector spaces, E and F, then

$$\mu\Big(\Big(\bigwedge^p f^\top\Big)(\omega)\Big)(u_1, \ldots, u_p) = \mu(\omega)(f(u_1), \ldots, f(u_p)), \qquad \omega \in \bigwedge^p F^*,\ \ u_1, \ldots, u_p \in E.$$

We apply this proposition with $E = \mathbb{R}^n$, $F = \mathbb{R}^m$, and $f = \varphi'_x$ $(x \in U)$, and get

$$\mu\Big(\Big(\bigwedge^p (\varphi'_x)^\top\Big)(\omega_{\varphi(x)})\Big)(u_1, \ldots, u_p) = \mu(\omega_{\varphi(x)})(\varphi'_x(u_1), \ldots, \varphi'_x(u_p)), \qquad \omega \in \mathcal{A}^p(V),\ \ u_i \in \mathbb{R}^n.$$

This gives us the behavior of $\bigwedge^p (\varphi'_x)^\top$ under the identification of $\bigwedge^p (\mathbb{R})^*$ and $\mathrm{Alt}^n(\mathbb{R}^n; \mathbb{R})$ *via* the isomorphism μ. Consequently, denoting $\bigwedge^p (\varphi'_x)^\top$ by φ^*, we make the following definition:

Definition 8.5 Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be two open subsets. For every smooth map, $\varphi \colon U \to V$, for every $p \geq 0$, we define the map, $\varphi^* \colon \mathcal{A}^p(V) \to \mathcal{A}^p(U)$, by

$$\varphi^*(\omega)_x(u_1, \ldots, u_p) = \omega_{\varphi(x)}(\varphi'_x(u_1), \ldots, \varphi'_x(u_p)),$$

for all $\omega \in \mathcal{A}^p(V)$, all $x \in U$ and all $u_1, \ldots, u_p \in \mathbb{R}^n$. We say that $\varphi^*(\omega)$ (for short, $\varphi^*\omega$) is the *pull-back of ω by φ*.

As φ is smooth, $\varphi^*\omega$ is a smooth p-form on U. The maps $\varphi^* \colon \mathcal{A}^p(V) \to \mathcal{A}^p(U)$ induce a map also denoted $\varphi^* \colon \mathcal{A}^*(V) \to \mathcal{A}^*(U)$. Using the chain rule, we check immediately that

$$\begin{aligned} \mathrm{id}^* &= \mathrm{id}, \\ (\psi \circ \varphi)^* &= \varphi^* \circ \psi^*. \end{aligned}$$

As an example, consider the constant form, $\omega = e_i^*$. We claim that $\varphi^* e_i^* = d\varphi_i$, where $\varphi_i = pr_i \circ \varphi$. Indeed,

$$\begin{aligned}
(\varphi^* e_i^*)_x(u) &= e_i^*(\varphi'_x(u)) \\
&= e_i^*\left(\sum_{k=1}^m \left(\sum_{l=1}^n \frac{\partial \varphi_k}{\partial x_l}(x)\, u_l\right) e_k\right) \\
&= \sum_{l=1}^n \frac{\partial \varphi_i}{\partial x_l}(x)\, u_l \\
&= \sum_{l=1}^n \frac{\partial \varphi_i}{\partial x_l}(x)\, e_l^*(u) = d(\varphi_i)_x(u).
\end{aligned}$$

For another example, assume U and V are open subsets of $\mathbb{R}^n$, $\omega = f dx_1 \wedge \cdots \wedge dx_n$, and write $x = \varphi(y)$, with x coordinates on V and y coordinates on U. Then

$$(\varphi^*\omega)_y = f(\varphi(y)) \det\left(\frac{\partial \varphi_i}{\partial y_j}(y)\right) dy_1 \wedge \cdots \wedge dy_p = f(\varphi(y)) J(\varphi)_y\, dy_1 \wedge \cdots \wedge dy_p,$$

where

$$J(\varphi)_y = \det\left(\frac{\partial \varphi_i}{\partial y_j}(y)\right)$$

is the Jacobian of φ at $y \in U$.

Proposition 8.6 *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be two open sets and let $\varphi\colon U \to V$ be a smooth map. Then*

(i) $\varphi^*(\omega \wedge \eta) = \varphi^*\omega \wedge \varphi^*\eta$, *for all* $\omega \in \mathcal{A}^p(V)$ *and all* $\eta \in \mathcal{A}^q(V)$.

(ii) $\varphi^*(f) = f \circ \varphi$, *for all* $f \in \mathcal{A}^0(V)$.

(iii) $d\varphi^*(\omega) = \varphi^*(d\omega)$, *for all* $\omega \in \mathcal{A}^p(V)$, *that is, the following diagram commutes for all* $p \geq 0$*:*

$$\begin{tikzcd}
\mathcal{A}^p(V) \arrow[r, "\varphi^*"] \arrow[d, "d"'] & \mathcal{A}^p(U) \arrow[d, "d"] \\
\mathcal{A}^{p+1}(V) \arrow[r, "\varphi^*"] & \mathcal{A}^{p+1}(U).
\end{tikzcd}$$

Proof. We leave the proof of (i) and (ii) as an exercise (or see Madsen and Tornehave [100], Chapter 3). First, we prove (iii) in the case $\omega \in \mathcal{A}^0(V)$. Using (i) and (ii) and the calculation just before Proposition 8.6, we have

$$\begin{aligned}
\varphi^*(df) &= \sum_{k=1}^{m} \varphi^*\left(\frac{\partial f}{\partial x_k}\right) \wedge \varphi^*(e_k^*) \\
&= \sum_{k=1}^{m} \left(\frac{\partial f}{\partial x_k} \circ \varphi\right) \wedge \left(\sum_{l=1}^{n} \frac{\partial \varphi_k}{\partial x_l}\, e_l^*\right) \\
&= \sum_{k=1}^{m} \sum_{l=1}^{n} \left(\frac{\partial f}{\partial x_k} \circ \varphi\right) \left(\frac{\partial \varphi_k}{\partial x_l}\right) e_l^* \\
&= \sum_{l=1}^{n} \left(\sum_{k=1}^{m} \left(\frac{\partial f}{\partial x_k} \circ \varphi\right) \frac{\partial \varphi_k}{\partial x_l}\right) e_l^* \\
&= \sum_{l=1}^{n} \frac{\partial (f \circ \varphi)}{\partial x_l}\, e_l^* \\
&= d(f \circ \varphi) = d(\varphi^*(f)).
\end{aligned}$$

For the case where $\omega = f e_I^*$, we know that $d\omega = df \wedge e_I^*$. We claim that

$$d\varphi^*(e_I^*) = 0.$$

This is because

$$\begin{aligned}
d\varphi^*(e_I^*) &= d(\varphi^*(e_{i_1}^*) \wedge \cdots \wedge \varphi^*(e_{i_p}^*)) \\
&= \sum (-1)^{k-1} \varphi^*(e_{i_1}^*) \wedge \cdots \wedge d(\varphi^*(e_{i_k}^*)) \wedge \cdots \wedge \varphi^*(e_{i_p}^*) = 0,
\end{aligned}$$

since $\varphi^*(e_{i_k}^*) = d\varphi_{i_k}$ and $d \circ d = 0$. Consequently,

$$d(\varphi^*(f) \wedge \varphi^*(e_I^*)) = d(\varphi^* f) \wedge \varphi^*(e_I^*).$$

Then, we have

$$\varphi^*(d\omega) = \varphi^*(df) \wedge \varphi^*(e_I^*) = d(\varphi^* f) \wedge \varphi^*(e_I^*) = d(\varphi^*(f) \wedge \varphi^*(e_I^*)) = d(\varphi^*(f e_I^*)) = d(\varphi^* \omega).$$

Since every differential form is a linear combination of special forms, $f e_I^*$, we are done. $\square$

The fact that d and pull-back commutes is an important fact: It allows us to show that a map, $\varphi \colon U \to V$, induces a map, $H^\bullet(\varphi) \colon H^\bullet(V) \to H^\bullet(U)$, on cohomology and it is crucial in generalizing the exterior differential to manifolds.

To a smooth map, $\varphi \colon U \to V$, we associate the map, $H^p(\varphi) \colon H^p(V) \to H^p(U)$, given by

$$H^p(\varphi)([\omega]) = [\varphi^*(\omega)].$$

This map is well defined because if we pick any representative, $\omega + d\eta$ in the cohomology class, $[\omega]$, specified by the closed form, ω, then

$$d\varphi^* \omega = \varphi^* d\omega = 0$$

so $\varphi^*\omega$ is closed and

$$\varphi^*(\omega + d\eta) = \varphi^*\omega + \varphi^*(d\eta) = \varphi^*\omega + d\varphi^*\eta,$$

so $H^p(\varphi)([\omega])$ is well defined. It is also clear that

$$H^{p+q}(\varphi)([\omega][\eta]) = H^p(\varphi)([\omega])H^q(\varphi)([\eta]),$$

which means that $H^\bullet(\varphi)$ is a homomorphism of graded algebras. We often denote $H^\bullet(\varphi)$ again by φ^*.

We conclude this section by stating without proof an important result known as the *Poincaré Lemma*. Recall that a subset, $S \subseteq \mathbb{R}^n$ is *star-shaped* iff there is some point, $c \in S$, such that for every point, $x \in S$, the closed line segment, $[c, x]$, joining c and x is entirely contained in S.

Theorem 8.7 *(Poincaré's Lemma)* If $U \subseteq \mathbb{R}^n$ is any star-shaped open set, then we have $H^p(U) = (0)$ for $p > 0$ and $H^0(U) = \mathbb{R}$. Thus, for every $p \geq 1$, every closed form $\omega \in \mathcal{A}^p(U)$ is exact.

Proof. Pick c so that U is star-shaped w.r.t. c and let $g\colon U \to U$ be the constant function with value c. Then, we see that

$$g^*\omega = \begin{cases} 0 & \text{if } \omega \in \mathcal{A}^p(U), \text{ with } p \geq 1, \\ \omega(c) & \text{if } \omega \in \mathcal{A}^0(U), \end{cases}$$

where $\omega(c)$ denotes the constant function with value $\omega(c)$. The trick is to find a family of linear maps, $h^p\colon \mathcal{A}^p(U) \to \mathcal{A}^{p-1}(U)$, for $p \geq 1$, with $h^0 = 0$, such that

$$d \circ h^p + h^{p+1} \circ d = \mathrm{id} - g^*, \qquad p > 0$$

called a *chain homotopy*. Indeed, if $\omega \in \mathcal{A}^p(U)$ is closed and $p \geq 1$, we get $dh^p\omega = \omega$, so ω is exact and if $p = 0$, we get $h^1 d\omega = 0 = \omega - \omega(c)$, so ω is constant. It remains to find the h^p, which is not obvious. A construction of these maps can be found in Madsen and Tornehave [100] (Chapter 3), Warner [147] (Chapter 4), Cartan [30] (Section 2) Morita [114] (Chapter 3). $\square$

In Section 8.2, we promote differential forms to manifolds. As preparation, note that every open subset, $U \subseteq \mathbb{R}^n$, is a manifold and that for every $x \in U$ the tangent space, T_xU, to U at x is canonically isomorphic to $\mathbb{R}^n$. It follows that the tangent bundle, TU, and the cotangent bundle, T^*U, are trivial, namely, $TU \cong U \times \mathbb{R}^n$ and $T^*U \cong U \times (\mathbb{R}^n)^*$, so the bundle,

$$\bigwedge^p T^*U \cong U \times \bigwedge^p (\mathbb{R}^n)^*,$$

is also trivial. Consequently, we can view $\mathcal{A}^p(U)$ as the set of smooth sections of the vector bundle, $\bigwedge^p T^*(U)$. The generalization to manifolds is then to define the space of differential p-forms on a manifold, M, as the space of smooth sections of the bundle, $\bigwedge^p T^*M$.

8.2 Differential Forms on Manifolds

Let M be any smooth manifold of dimension n. We define the vector bundle, $\bigwedge T^*M$, as the direct sum bundle,

$$\bigwedge T^*M = \bigoplus_{p=0}^{n} \bigwedge^p T^*M,$$

see Section 7.3 for details.

Definition 8.6 Let M be any smooth manifold of dimension n. The set, $\mathcal{A}^p(M)$, of *smooth differential p-forms* on M is the set of smooth sections, $\Gamma(M, \bigwedge^p T^*M)$, of the bundle $\bigwedge^p T^*M$ and the set, $\mathcal{A}^*(M)$, of all *smooth differential forms* on M is the set of smooth sections, $\Gamma(M, \bigwedge T^*M)$, of the bundle $\bigwedge T^*M$.

Observe that $\mathcal{A}^0(M) \cong C^\infty(M, \mathbb{R})$, the set of smooth functions on M, since the bundle $\bigwedge^0 T^*M$ is isomorphic to $M \times \mathbb{R}$ and smooth sections of $M \times \mathbb{R}$ are just graphs of smooth functions on M. We also write $C^\infty(M)$ for $C^\infty(M, \mathbb{R})$. If $\omega \in \mathcal{A}^*(M)$, we often write ω_x for $\omega(x)$.

Definition 8.6 is quite abstract and it is important to get a more down-to-earth feeling by taking a local view of differential forms, namely, with respect to a chart. So, let (U, φ) be a local chart on M, with $\varphi\colon U \to \mathbb{R}^n$, and let $x_i = pr_i \circ \varphi$, the ith local coordinate ($1 \leq i \leq n$) (see Section 3.2). Recall that by Proposition 3.4, for any $p \in U$, the vectors

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_x}\right)_p$$

form a basis of the tangent space, T_pM. Furthermore, by Proposition 3.9 and the discussion following Proposition 3.8, the linear forms, $(dx_1)_p, \ldots, (dx_n)_p$ form a basis of T_p^*M, (where $(dx_i)_p$, the differential of x_i at p, is identified with the linear form such that $df_p(v) = v(\mathbf{f})$, for every smooth function f on U and every $v \in T_pM$). Consequently, locally on U, every k-form, $\omega \in \mathcal{A}^k(M)$, can be written uniquely as

$$\omega = \sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \sum_I f_I dx_I, \qquad p \in U,$$

where $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, with $i_1 < \ldots < i_k$ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Furthermore, each f_I is a smooth function on U.

Remark: We define the set of smooth (r, s)-*tensor fields* as the set, $\Gamma(M, T^{r,s}(M))$, of smooth sections of the tensor bundle $T^{r,s}(M) = T^{\otimes r}M \otimes (T^*M)^{\otimes s}$. Then, locally in a chart (U, φ), every tensor field $\omega \in \Gamma(M, T^{r,s}(M))$ can be written uniquely as

$$\omega = \sum f^{i_1, \ldots, i_r}_{j_1, \ldots, j_s} \left(\frac{\partial}{\partial x_{i_1}}\right) \otimes \cdots \otimes \left(\frac{\partial}{\partial x_{i_r}}\right) \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}.$$

The operations on the algebra, $\bigwedge T^*M$, yield operations on differential forms using pointwise definitions. If $\omega, \eta \in \mathcal{A}^*(M)$ and $\lambda \in \mathbb{R}$, then for every $x \in M$,

$$\begin{aligned}(\omega+\eta)_x &= \omega_x + \eta_x\\ (\lambda\omega)_x &= \lambda\omega_x\\ (\omega\wedge\eta)_x &= \omega_x \wedge \eta_x.\end{aligned}$$

Actually, it is necessary to check that the resulting forms are smooth but this is easily done using charts. When, $f \in \mathcal{A}^0(M)$, we write $f\omega$ instead of $f \wedge \omega$. It follows that $\mathcal{A}^*(M)$ is a graded real algebra and a $C^\infty(M)$-module.

Proposition 8.1 generalizes immediately to manifolds.

Proposition 8.8 *For all forms $\omega \in \mathcal{A}^r(M)$ and $\eta \in \mathcal{A}^s(M)$, we have*

$$\eta \wedge \omega = (-1)^{pq}\omega \wedge \eta.$$

For any smooth map, $\varphi\colon M \to N$, between two manifolds, M and N, we have the differential map, $d\varphi\colon TM \to TN$, also a smooth map and, for every $p \in M$, the map $d\varphi_p\colon T_pM \to T_{\varphi(p)}N$ is linear. As in Section 8.1, Proposition 22.21 gives us the formula

$$\mu\Big(\Big(\bigwedge^k (d\varphi_p)^\top\Big)(\omega_{\varphi(p)})\Big)(u_1,\ldots,u_k) = \mu(\omega_{\varphi(p)})(d\varphi_p(u_1),\ldots,d\varphi_p(u_k)), \qquad \omega \in \mathcal{A}^k(N),$$

for all $u_1,\ldots,u_k \in T_pM$. This gives us the behavior of $\bigwedge^k (d\varphi_p)^\top$ under the identification of $\bigwedge^k T_p^*M$ and $\mathrm{Alt}^k(T_pM;\mathbb{R})$ *via* the isomorphism μ. Here is the extension of Definition 8.5 to differential forms on a manifold.

Definition 8.7 For any smooth map, $\varphi\colon M \to N$, between two smooth manifolds, M and N, for every $k \geq 0$, we define the map, $\varphi^*\colon \mathcal{A}^k(N) \to \mathcal{A}^k(M)$, by

$$\varphi^*(\omega)_p(u_1,\ldots,u_k) = \omega_{\varphi(p)}(d\varphi_p(u_1),\ldots,d\varphi_p(u_k)),$$

for all $\omega \in \mathcal{A}^k(N)$, all $p \in M$, and all $u_1,\ldots,u_k \in T_pM$. We say that $\varphi^*(\omega)$ (for short, $\varphi^*\omega$) is the *pull-back of ω by φ*.

The maps $\varphi^*\colon \mathcal{A}^k(N) \to \mathcal{A}^k(M)$ induce a map also denoted $\varphi^*\colon \mathcal{A}^*(N) \to \mathcal{A}^*(M)$. Using the chain rule, we check immediately that

$$\begin{aligned}\mathrm{id}^* &= \mathrm{id},\\ (\psi\circ\varphi)^* &= \varphi^*\circ\psi^*.\end{aligned}$$

We need to check that $\varphi^*\omega$ is smooth and for this, it is enough to check it locally on a chart, (U,φ). On U, we know that $\omega \in \mathcal{A}^k(M)$ can be written uniquely as

$$\omega = \sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \qquad p \in U,$$

with f_I smooth and it is easy to see (using the definition) that

$$\varphi^*\omega = \sum_I (f_I \circ \varphi) d(x_{i_1} \circ \varphi) \wedge \cdots \wedge d(x_{i_k} \circ \varphi),$$

which is smooth.

Remark: The fact that the pull-back of differential forms makes sense for arbitrary smooth maps, $\varphi\colon M \to N$, and not just diffeomorphisms is a major technical superiority of forms over vector fields.

The next step is to define d on $\mathcal{A}^*(M)$. There are several ways to proceed but since we already considered the special case where M is an open subset of $\mathbb{R}^n$, we proceed using charts.

Given a smooth manifold, M, of dimension n, let (U, φ) be any chart on M. For any $\omega \in \mathcal{A}^k(M)$ and any $p \in U$, define $(d\omega)_p$ as follows: If $k = 0$, that is, $\omega \in C^\infty(M)$, let

$$(d\omega)_p = d\omega_p, \qquad \text{the differential of } \omega \text{ at } p$$

and if $k \geq 1$, let

$$(d\omega)_p = \varphi^*\big(d((\varphi^{-1})^*\omega)_{\varphi(p)}\big)_p,$$

where d is the exterior differential on $\mathcal{A}^k(\varphi(U))$. More explicitly, $(d\omega)_p$ is given by

$$(d\omega)_p(u_1, \ldots, u_{k+1}) = d((\varphi^{-1})^*\omega)_{\varphi(p)}(d\varphi_p(u_1), \ldots, d\varphi_p(u_{k+1})),$$

for every $p \in U$ and all $u_1, \ldots, u_{k+1} \in T_pM$. Observe that the above formula is still valid when $k = 0$ if we interpret the symbold d in $d((\varphi^{-1})^*\omega)_{\varphi(p)} = d(\omega \circ \varphi^{-1})_{\varphi(p)}$ as the differential.

Since $\varphi^{-1}\colon \varphi(U) \to U$ is map whose domain is an open subset, $W = \varphi(U)$, of $\mathbb{R}^n$, the form $(\varphi^{-1})^*\omega$ is a differential form in $\mathcal{A}^*(W)$, so $d((\varphi^{-1})^*\omega)$ is well-defined. We need to check that this definition does not depend on the chart, (U, φ). For any other chart, (V, ψ), with $U \cap V \neq \emptyset$, the map $\theta = \psi \circ \varphi^{-1}$ is a diffeomorphism between the two open subsets, $\varphi(U \cap V)$ and $\psi(U \cap V)$, and $\psi = \theta \circ \varphi$. Let $x = \varphi(p)$. We need to check that

$$d((\varphi^{-1})^*\omega)_x(d\varphi_p(u_1), \ldots, d\varphi_p(u_{k+1})) = d((\psi^{-1})^*\omega)_x(d\psi_p(u_1), \ldots, d\psi_p(u_{k+1})),$$

for every $p \in U \cap V$ and all $u_1, \ldots, u_{k+1} \in T_pM$. However,

$$d((\psi^{-1})^*\omega)_x(d\psi_p(u_1), \ldots, d\psi_p(u_{k+1})) = d((\varphi^{-1} \circ \theta^{-1})^*\omega)_x(d(\theta \circ \varphi)_p(u_1), \ldots, d(\theta \circ \varphi)_p(u_{k+1})),$$

and since

$$(\varphi^{-1} \circ \theta^{-1})^* = (\theta^{-1})^* \circ (\varphi^{-1})^*$$

and, by Proposition 8.6 (iii),

$$d(((\theta^{-1})^* \circ (\varphi^{-1})^*)\omega) = d((\theta^{-1})^*((\varphi^{-1})^*\omega)) = (\theta^{-1})^*(d((\varphi^{-1})^*\omega)),$$

we get

$$\begin{aligned}
&d((\varphi^{-1}\circ\theta^{-1})^*\omega)_x(d(\theta\circ\varphi)_p(u_1),\ldots,d(\theta\circ\varphi)_p(u_{k+1}))\\
&\qquad\qquad = (\theta^{-1})^*(d((\varphi^{-1})^*\omega))_{\theta(x)}(d(\theta\circ\varphi)_p(u_1),\ldots,d(\theta\circ\varphi)_p(u_{k+1}))
\end{aligned}$$

and then

$$\begin{aligned}
&(\theta^{-1})^*(d((\varphi^{-1})^*\omega))_{\theta(x)}(d(\theta\circ\varphi)_p(u_1),\ldots,d(\theta\circ\varphi)_p(u_{k+1}))\\
&\qquad = d((\varphi^{-1})^*\omega)_x((d\theta^{-1})_{\theta(x)}(d(\theta\circ\varphi)_p(u_1)),\ldots,(d\theta^{-1})_{\theta(x)}(d(\theta\circ\varphi)_p(u_{k+1}))).
\end{aligned}$$

As $(d\theta^{-1})_{\theta(x)}(d(\theta\circ\varphi)_p(u_1)) = d(\theta^{-1}\circ(\theta\circ\varphi))_p(u_i) = d\varphi_p(u_i)$, by the chain rule, we obtain

$$d((\psi^{-1})^*\omega)_x(d\psi_p(u_1),\ldots,d\psi_p(u_{k+1})) = d((\varphi^{-1})^*\omega)_x(d\varphi_p(u_1),\ldots,d\varphi_p(u_{k+1})),$$

as desired.

Observe that $(d\omega)_p$ is smooth on U and as our definition of $(d\omega)_p$ does not depend on the choice of a chart, the forms $(d\omega)\restriction U$ agree on overlaps and yield a differential form, $d\omega$, defined on the whole of M. Thus, we can make the following definition:

Definition 8.8 If M is any smooth manifold, there is a linear map, $d\colon \mathcal{A}^k(M)\to\mathcal{A}^{k+1}(M)$, for every $k\geq 0$, such that, for every $\omega\in\mathcal{A}^k(M)$, for every chart, (U,φ), for every $p\in U$, if $k=0$, that is, $\omega\in C^\infty(M)$, then

$$(d\omega)_p = d\omega_p, \qquad \text{the differential of } \omega \text{ at } p,$$

else if $k\geq 1$, then

$$(d\omega)_p = \varphi^*\big(d((\varphi^{-1})^*\omega)_{\varphi(p)}\big)_p,$$

where d is the exterior differential on $\mathcal{A}^k(\varphi(U))$ from Definition 8.3. We obtain a linar map, $d\colon \mathcal{A}^*(M)\to\mathcal{A}^*(M)$, called *exterior differentiation*.

Propositions 8.3, 8.4 and 8.6 generalize to manifolds.

Proposition 8.9 *Let M and N be smooth manifolds and let $\varphi\colon M\to N$ be a smooth map.*

(1) For all $\omega\in\mathcal{A}^r(M)$ and all $\eta\in\mathcal{A}^s(M)$,

$$d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^r\omega\wedge d\eta.$$

(2) For every $k\geq 0$, the composition $\mathcal{A}^k(M)\xrightarrow{d}\mathcal{A}^{k+1}(M)\xrightarrow{d}\mathcal{A}^{k+2}(M)$ is identically zero, that is,

$$d\circ d = 0.$$

(3) $\varphi^*(\omega\wedge\eta) = \varphi^*\omega\wedge\varphi^*\eta$, for all $\omega\in\mathcal{A}^r(N)$ and all $\eta\in\mathcal{A}^s(N)$.

(4) $\varphi^*(f) = f \circ \varphi$, *for all* $f \in \mathcal{A}^0(N)$.

(5) $d\varphi^*(\omega) = \varphi^*(d\omega)$, *for all* $\omega \in \mathcal{A}^k(N)$, *that is, the following diagram commutes for all* $k \geq 0$*:*

$$\begin{tikzcd}
\mathcal{A}^k(N) \arrow[r, "\varphi^*"] \arrow[d, "d"'] & \mathcal{A}^k(M) \arrow[d, "d"] \\
\mathcal{A}^{k+1}(N) \arrow[r, "\varphi^*"] & \mathcal{A}^{k+1}(M).
\end{tikzcd}$$

Proof. It is enough to prove these properties in a chart, (U, φ), which is easy. We only check (2). We have

$$\begin{aligned}
(d(d\omega))_p &= d\big(\varphi^*\big(d((\varphi^{-1})^*\omega)\big)\big)_p \\
&= \varphi^*\Big[d(\varphi^{-1})^*\big(\varphi^*\big(d((\varphi^{-1})^*\omega)\big)\big)_{\varphi(p)}\Big]_p \\
&= \varphi^*\Big[d\big(d((\varphi^{-1})^*\omega)\big)_{\varphi(p)}\Big]_p \\
&= 0,
\end{aligned}$$

as $(\varphi^{-1})^* \circ \varphi^* = (\varphi \circ \varphi^{-1})^* = \mathrm{id}^* = \mathrm{id}$ and $d \circ d = 0$ on forms in $\mathcal{A}^k(\varphi(U))$, with $\varphi(U) \subseteq \mathbb{R}^n$. $\square$

As a consequence, Definition 8.4 of the de Rham cohomology generalizes to manifolds. For every manifold, M, we have the de Rham complex,

$$\mathcal{A}^0(M) \xrightarrow{d} \mathcal{A}^1(M) \longrightarrow \cdots \longrightarrow \mathcal{A}^{k-1}(M) \xrightarrow{d} \mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M) \longrightarrow \cdots$$

and we can define the *cohomology groups*, $H^k_{\mathrm{DR}}(M)$, and the graded *cohomology algebra*, $H^\bullet_{\mathrm{DR}}(M)$. For every $k \geq 0$, let

$$Z^k(M) = \{\omega \in \mathcal{A}^k(M) \mid d\omega = 0\} = \mathrm{Ker}\, d\colon \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{k+1}(M),$$

be the vector space of closed k-forms and for every $k \geq 1$, let

$$B^k(M) = \{\omega \in \mathcal{A}^k(M) \mid \exists \eta \in \mathcal{A}^{k-1}(M),\ \omega = d\eta\} = \mathrm{Im}\, d\colon \mathcal{A}^{k-1}(M) \longrightarrow \mathcal{A}^k(M),$$

be the vector space of exact k-forms and set $B^0(M) = (0)$. Then, for every $k \geq 0$, we define the k^{th} *de Rham cohomology group* of M as the quotient space

$$H^k_{\mathrm{DR}}(M) = Z^k(M)/B^k(M).$$

The real vector space, $H^\bullet_{\mathrm{DR}}(M) = \bigoplus_{k\geq 0} H^k_{\mathrm{DR}}(M)$, is called the *de Rham cohomology algebra* of M. We often drop the subscript, DR, when no confusion arises. Every smooth map, $\varphi\colon M \to N$, between two manifolds induces an algebra map, $\varphi^*\colon H^\bullet(N) \to H^\bullet(M)$.

Another important property of the exterior differential is that it is a *local operator*, which means that the value of $d\omega$ at p only depends of the values of ω near p. More precisely, we have

Proposition 8.10 *Let M be a smooth manifold. For every open subset, $U \subseteq M$, for any two differential forms, $\omega, \eta \in \mathcal{A}^*(M)$, if $\omega \restriction U = \eta \restriction U$, then $(d\omega) \restriction U = (d\eta) \restriction U$.*

Proof. By linearity, it is enough to show that if $\omega \restriction U = 0$, then $(d\omega) \restriction U = 0$. The crucial ingredient is the existence of "bump functions". By Proposition 3.24 applied to the constant function with value 1, for every $p \in U$, there some open subset, $V \subseteq U$, containing p and a smooth function, $f\colon M \to \mathbb{R}$, such that $\operatorname{supp} f \subseteq U$ and $f \equiv 1$ on V. Consequently, $f\omega$ is a smooth differential form which is identically zero and by Proposition 8.9 (1),

$$d(f\omega) = df \wedge \omega + f d\omega,$$

which, evaluated ap p, yields

$$0 = 0 \wedge \omega_p + 1 d\omega_p,$$

that is, $d\omega_p = 0$, as claimed. $\square$

As in the case of differential forms on $\mathbb{R}^n$, the operator d is uniquely determined by the properties of Theorem 8.5.

Theorem 8.11 *Let M be a smooth manifold. There is a unique local linear map, $d\colon \mathcal{A}^*(M) \to \mathcal{A}^*(M)$, with $d = (d^k)$ and $d^k\colon \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ for every $k \geq 0$, such that*

(1) $(df)_p = df_p$, where df_p is the differential of f at $p \in M$, for every $f \in \mathcal{A}^0(M) = C^\infty(M)$.

(2) $d \circ d = 0$.

(3) For every $\omega \in \mathcal{A}^r(M)$ and every $\eta \in \mathcal{A}^s(M)$,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta.$$

Proof. Existence has already been established. It is enough to prove uniqueness locally. If (U, φ) is any chart and $x_i = pr_i \circ \varphi$ are the corresponding local coordinate maps, we know that every k-form, $\omega \in \mathcal{A}^k(M)$, can be written uniquely as

$$\omega = \sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k} \qquad p \in U.$$

Consequently, the proof of Theorem 8.5 will go through if we can show that $ddx_{i_j} \restriction U = 0$, since then,

$$d(f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

The problem is that dx_{i_j} is only defined on U. However, using Proposition 3.24 again, for every $p \in U$, there some open subset, $V \subseteq U$, containing p and a smooth function, $f\colon M \to \mathbb{R}$, such that $\operatorname{supp} f \subseteq U$ and $f \equiv 1$ on V. Then, $f x_{i_j}$ is a smooth form defined on M such that $f x_{i_j} \restriction V = x_{i_j} \restriction V$, so by Proposition 8.10 (applied twice),

$$0 = dd(f x_{i_j}) \restriction V = ddx_{i_j} \restriction V,$$

which concludes the proof. $\square$

Remark: A closer look at the proof of Theorem 8.11 shows that it is enough to assume $dd\omega = 0$ on forms $\omega \in \mathcal{A}^0(M) = C^\infty(M)$.

Smooth differential forms can also be defined in terms of alternating $C^\infty(M)$-multilinear maps on smooth vector fields. Let $\omega \in \mathcal{A}^p(M)$ be any smooth k-form on M. Then, ω induces an alternating multilinear map

$$\omega\colon \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \longrightarrow C^\infty(M)$$

as follows: For any k smooth vector fields, $X_1, \ldots, X_k \in \mathfrak{X}(M)$,

$$\omega(X_1, \ldots, X_k)(p) = \omega_p(X_1(p), \ldots, X_k(p)).$$

This map is obviously alternating and $\mathbb{R}$-linear, but it is also $C^\infty(M)$-linear, since for every $f \in C^\infty(M)$,

$$\begin{aligned} \omega(X_1, \ldots, fX_i, \ldots X_k)(p) &= \omega_p(X_1(p), \ldots, f(p)X_i(p), \ldots, X_k(p)) \\ &= f(p)\omega_p(X_1(p), \ldots, X_i(p), \ldots, X_k(p)) \\ &= (f\omega)_p(X_1(p), \ldots, X_i(p), \ldots, X_k(p)). \end{aligned}$$

(Recall, that the set of smooth vector fields, $\mathfrak{X}(M)$, is a real vector space and a $C^\infty(M)$-module.)

Interestingly, every alternating $C^\infty(M)$-multilinear maps on smooth vector fields determines a differential form. This is because $\omega(X_1, \ldots, X_k)(p)$ only depends on the values of $X_1, \ldots, X_k$ at p.

Proposition 8.12 *Let M be a smooth manifold. For every $k \geq 0$, there is an isomorphism between the space of k-forms, $\mathcal{A}^k(M)$, and the space, $\mathrm{Alt}^k_{C^\infty(M)}(\mathfrak{X}(M))$, of alternating $C^\infty(M)$-multilinear maps on smooth vector fields. That is,*

$$\mathcal{A}^k(M) \cong \mathrm{Alt}^k_{C^\infty(M)}(\mathfrak{X}(M)),$$

viewed as $C^\infty(M)$-modules.

Proof. Let $\Phi\colon \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \longrightarrow C^\infty(M)$ be an alternating $C^\infty(M)$-multilinear map. First, we prove that for any vector fields $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$, for every $p \in M$, if $X_i(p) = Y_i(p)$, then

$$\Phi(X_1, \ldots, X_k)(p) = \Phi(Y_1, \ldots, Y_k)(p).$$

Observe that

$$\begin{aligned}\Phi(X_1,\ldots,X_k)-\Phi(Y_1,\ldots,Y_k) &= \Phi(X_1-Y_1,X_2,\ldots,X_k)+\Phi(Y_1,X_2-Y_2,X_3,\ldots,X_k)\\ &= \quad +\Phi(Y_1,Y_2,X_3-Y_3,\ldots,X_k)+\cdots\\ &= \quad +\Phi(Y_1,\ldots,Y_{k-2},X_{k-1}-Y_{k-1},X_k)\\ &= \quad +\cdots+\Phi(Y_1,\ldots,Y_{k-1},X_k-Y_k).\end{aligned}$$

As a consequence, it is enough to prove that if $X_i(p)=0$, for some i, then

$$\Phi(X_1,\ldots,X_k)(p)=0.$$

Without loss of generality, assume $i=1$. In any local chart, (U,φ), near p, we can write

$$X_1=\sum_{i=1}^n f_i\,\frac{\partial}{\partial x_i},$$

and as $X_i(p)=0$, we have $f_i(p)=0$, for $i=1,\ldots,n$. Since the expression on the right-hand side is only defined on U, we extend it using Proposition 3.24, once again. There is some open subset, $V\subseteq U$, containing p and a smooth function, $h\colon M\to\mathbb{R}$, such that $\operatorname{supp} h\subseteq U$ and $h\equiv 1$ on V. Then, we let $h_i=hf_i$, a smooth function on M, $Y_i=h\frac{\partial}{\partial x_i}$, a smooth vector field on M, and we have $h_i\upharpoonright V=f_i\upharpoonright V$ and $Y_i\upharpoonright V=\frac{\partial}{\partial x_i}\upharpoonright V$. Now, it it obvious that

$$X_1=\sum_{i=1}^n h_iY_i+(1-h^2)X_1,$$

so, as Φ is $C^\infty(M)$-multilinear, $h_i(p)=0$ and $h(p)=1$, we get

$$\begin{aligned}\Phi(X_1,X_2,\ldots,X_k)(p) &= \Phi(\sum_{i=1}^n h_iY_i+(1-h^2)X_1,X_2,\ldots,X_k)(p)\\ &= \sum_{i=1}^n h_i(p)\Phi(Y_i,X_2,\ldots,X_k)(p)+(1-h^2(p))\Phi(X_1,X_2,\ldots,X_k)(p)=0,\end{aligned}$$

as claimed.

Next, we show that Φ induces a smooth differential form. For every $p\in M$, for any $u_1,\ldots,u_k\in T_pM$, we can pick smooth functions, f_i, equal to 1 near p and 0 outside some open near p so that we get smooth vector fields, $X_1,\ldots,X_k$, with $X_k(p)=u_k$. We set

$$\omega_p(u_1,\ldots,u_k)=\Phi(X_1,\ldots,X_k)(p).$$

As we proved that $\Phi(X_1,\ldots,X_k)(p)$ only depends on $X_1(p)=u_1,\ldots,X_k(p)=u_k$, the function ω_p is well defined and it is easy to check that it is smooth. Therefore, the map, $\Phi\mapsto\omega$, just defined is indeed an isomorphism. $\square$

Remarks:

(1) The space, $\mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M))$, of all $C^\infty(M)$-linear maps, $\mathfrak{X}(M) \longrightarrow C^\infty(M)$, is also a $C^\infty(M)$-module called the *dual* of $\mathfrak{X}(M)$ and sometimes denoted $\mathfrak{X}^*(M)$. Proposition 8.12 shows that as $C^\infty(M)$-modules,

$$\mathcal{A}^1(M) \cong \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M)) = \mathfrak{X}^*(M).$$

(2) A result analogous to Proposition 8.12 holds for tensor fields. Indeed, there is an isomorphism between the set of tensor fields, $\Gamma(M, T^{r,s}(M))$, and the set of $C^\infty(M)$-multilinear maps,

$$\Phi\colon \underbrace{\mathcal{A}^1(M) \times \cdots \times \mathcal{A}^1(M)}_{r} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s} \longrightarrow C^\infty(M),$$

where $\mathcal{A}^1(M)$ and $\mathfrak{X}(M)$ are $C^\infty(M)$-modules.

Recall from Section 3.3 (Definition 3.15) that for any function, $f \in C^\infty(M)$, and every vector field, $X \in \mathfrak{X}(M)$, the Lie derivative, $X[f]$ (or $X(f)$) of f w.r.t. X is defined so that

$$X[f]_p = df_p(X(p)).$$

Also recall the notion of the *Lie bracket*, $[X, Y]$, of two vector fields (see Definition 3.16). The interpretation of differential forms as $C^\infty(M)$-multilinear forms given by Proposition 8.12 yields the following formula for $(d\omega)(X_1, \ldots, X_{k+1})$, where the X_i are vector fields:

Proposition 8.13 *Let M be a smooth manifold. For every k-form, $\omega \in \mathcal{A}^k(M)$, we have*

$$\begin{aligned}(d\omega)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} X_i[\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})] \\ + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1})],\end{aligned}$$

for all vector fields, $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$.

Proof sketch. First, one checks that the right-hand side of the formula in Proposition 8.13 is alternating and $C^\infty(M)$-multilinear. For this, use Proposition 3.13 (c). Consequently, by Proposition 8.12, this expression defines a $(k+1)$-form. Second, it is enough to check that both sides of the equation agree on charts, (U, φ). Then, we know that $d\omega$ can be written uniquely as

$$\omega = \sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k} \qquad p \in U.$$

Also, as differential forms are $C^\infty(M)$-multilinear, it is enough to consider vector fields of the form $X_i = \frac{\partial}{\partial x_{j_i}}$. However, for such vector fields, $[X_i, X_j] = 0$, and then it is a simple matter to check that the equation holds. For more details, see Morita [114] (Chapter 2). $\square$

In particular, when $k = 1$, Proposition 8.13 yields the often used formula:

$$d\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y]).$$

There are other ways of proving the formula of Proposition 8.13, for instance, using Lie derivatives.

Before considering the Lie derivative of differential forms, $L_X\omega$, we define interior multiplication by a vector field, $i(X)(\omega)$. We will see shortly that there is a relationship between L_X, $i(X)$ and d, known as *Cartan's Formula*.

Definition 8.9 Let M be a smooth manifold. For every vector field, $X \in \mathfrak{X}(M)$, for all $k \geq 1$, there is a linear map, $i(X)\colon \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$, defined so that, for all $\omega \in \mathcal{A}^k(M)$, for all $p \in M$, for all $u_1, \ldots, u_{k-1} \in T_pM$,

$$(i(X)\omega)_p(u_1, \ldots, u_{k-1}) = \omega_p(X_p, u_1, \ldots, u_{k-1}).$$

Obviously, $i(X)$ is $C^\infty(M)$-linear in X and it is easy to check that $i(X)\omega$ is indeed a smooth $(k-1)$-form. When $k = 0$, we set $i(X)\omega = 0$. Observe that $i(X)\omega$ is also given by

$$(i(X)\omega)_p = i(X_p)\omega_p, \qquad p \in M,$$

where $i(X_p)$ is the interior product (or insertion operator) defined in Section 22.17 (with $i(X_p)\omega_p$ equal to our right hook, $\omega_p \llcorner X_p$). As a consequence, by Proposition 22.28, the operator $i(X)$ is an anti-derivation of degree -1, that is, we have

$$i(X)(\omega \wedge \eta) = (i(X)\omega) \wedge \eta + (-1)^r \omega \wedge (i(X)\eta),$$

for all $\omega \in \mathcal{A}^r(M)$ and all $\eta \in \mathcal{A}^s(M)$.

Remark: Other authors, including Marsden, use a left hook instead of a right hook and denote $i(X)\omega$ as $X \lrcorner\, \omega$.

8.3 Lie Derivatives

We just saw in Section 8.2 that for any function, $f \in C^\infty(M)$, and every vector field, $X \in \mathfrak{X}(M)$, the Lie derivative, $X[f]$ (or $X(f)$) of f w.r.t. X is defined so that

$$X[f]_p = df_p(X_p).$$

Recall from Definition 3.24 and the observation immediately following it that for any manifold, M, given any two vector fields, $X, Y \in \mathfrak{X}(M)$, the *Lie derivative of X with respect to Y* is given by

$$(L_X\, Y)_p = \lim_{t \longrightarrow 0} \frac{\left(\Phi_t^* Y\right)_p - Y_p}{t} = \frac{d}{dt} \left(\Phi_t^* Y\right)_p \bigg|_{t=0},$$

where Φ_t is the local one-parameter group associated with X (Φ is the global flow associated with X, see Definition 3.23, Theorem 3.21 and the remarks following it) and Φ_t^* is the pull-back of the diffeomorphism Φ_t (see Definition 3.17). Furthermore, recall that

$$L_X Y = [X, Y].$$

We claim that we also have

$$X_p[f] = \lim_{t \longrightarrow 0} \frac{(\Phi_t^* f)(p) - f(p)}{t} = \left.\frac{d}{dt}(\Phi_t^* f)(p)\right|_{t=0},$$

with $\Phi_t^* f = f \circ \Phi_t$ (as usual for functions).

Recall from Section 3.5 that if Φ is the flow of X, then for every $p \in M$, the map, $t \mapsto \Phi_t(p)$, is an integral curve of X through p, that is

$$\dot{\Phi}_t(p) = X(\Phi_t(p)), \qquad \Phi_0(p) = p,$$

in some open set containing p. In particular, $\dot{\Phi}_0(p) = X_p$. Then, we have

$$\begin{aligned}
\lim_{t \longrightarrow 0} \frac{(\Phi_t^* f)(p) - f(p)}{t} &= \lim_{t \longrightarrow 0} \frac{f(\Phi_t(p)) - f(\Phi_0(p))}{t} \\
&= \left.\frac{d}{dt}(f \circ \Phi_t(p))\right|_{t=0} \\
&= df_p(\dot{\Phi}_0(p)) = df_p(X_p) = X_p[f].
\end{aligned}$$

We would like to define the Lie derivative of differential forms (and tensor fields). This can be done algebraically or in terms of flows, the two approaches are equivalent but it seems more natural to give a definition using flows.

Definition 8.10 Let M be a smooth manifold. For every vector field, $X \in \mathfrak{X}(M)$, for every k-form, $\omega \in \mathcal{A}^k(M)$, the *Lie derivative of ω with respect to X*, denoted $L_X\omega$ is given by

$$(L_X\omega)_p = \lim_{t \longrightarrow 0} \frac{\left(\Phi_t^*\omega\right)_p - \omega_p}{t} = \left.\frac{d}{dt}\left(\Phi_t^*\omega\right)_p\right|_{t=0},$$

where $\Phi_t^*\omega$ is the pull-back of ω along Φ_t (see Definition 8.7).

Obviously, $L_X \colon \mathcal{A}^k(M) \to \mathcal{A}^k(M)$ is a linear map but it has many other interesting properties. We can also define the Lie derivative on tensor fields as a map,
$L_X \colon \Gamma(M, T^{r,s}(M)) \to \Gamma(M, T^{r,s}(M))$, by requiring that for any tensor field,

$$\alpha = X_1 \otimes \cdots \otimes X_r \otimes \omega_1 \otimes \cdots \otimes \omega_s,$$

where $X_i \in \mathfrak{X}(M)$ and $\omega_j \in \mathcal{A}^1(M)$,

$$\Phi_t^* \alpha = \Phi_t^* X_1 \otimes \cdots \otimes \Phi_t^* X_r \otimes \Phi_t^* \omega_1 \otimes \cdots \otimes \Phi_t^* \omega_s,$$

where $\Phi_t^* X_i$ is the pull-back of the vector field, X_i, and $\Phi_t^* \omega_j$ is the pull-back of one-form, ω_j, and then setting

$$(L_X \alpha)_p = \lim_{t \longrightarrow 0} \frac{\left(\Phi_t^* \alpha\right)_p - \alpha_p}{t} = \frac{d}{dt} \left(\Phi_t^* \alpha\right)_p \Big|_{t=0}.$$

So, as long we can define the "right" notion of pull-back, the formula giving the Lie derivative of a function, a vector field, a differential form and more generally, a tensor field, is the same.

The Lie derivative of tensors is used in most areas of mechanics, for example in elasticity (the rate of strain tensor) and in fluid dynamics.

We now state, mostly without proofs, a number of properties of Lie derivatives. Most of these proofs are fairly straightforward computations, often tedious, and can be found in most texts, including Warner [147], Morita [114] and Gallot, Hullin and Lafontaine [60].

Proposition 8.14 *Let M be a smooth manifold. For every vector field, $X \in \mathfrak{X}(M)$, the following properties hold:*

(1) For all $\omega \in \mathcal{A}^r(M)$ and all $\eta \in \mathcal{A}^s(M)$,

$$L_X(\omega \wedge \eta) = (L_X \omega) \wedge \eta + \omega \wedge (L_X \eta),$$

that is, L_X is a derivation.

(2) For all $\omega \in \mathcal{A}^k(M)$, for all $Y_1, \ldots, Y_k \in \mathfrak{X}(M)$,

$$L_X(\omega(Y_1, \ldots, Y_k)) = (L_X \omega)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} \omega(Y_1, \ldots, Y_{i-1}, L_X Y_i, Y_{i+1}, \ldots, Y_k).$$

(3) The Lie derivative commutes with d:

$$L_X \circ d = d \circ L_X.$$

Proof. We only prove (2). First, we claim that if $\varphi \colon M \to M$ is a diffeomorphism, then for every $\omega \in \mathcal{A}^k(M)$, for all $X_1, \ldots, X_k \in \mathfrak{X}(M)$,

$$(\varphi^* \omega)(X_1, \ldots, X_k) = \varphi^*(\omega((\varphi^{-1})^* X_1, \ldots, (\varphi^{-1})^* X_k)), \tag{$*$}$$

where $(\varphi^{-1})^* X_i$ is the pull-back of the vector field, X_i (also equal to the push-forward, $\varphi_* X_i$, of X_i, see Definition 3.17). Recall that

$$((\varphi^{-1})^* Y)_p = d\varphi_{\varphi^{-1}(p)}(Y_{\varphi^{-1}(p)}),$$

for any vector field, Y. Then, for every $p \in M$, we have

$$\begin{aligned}
(\varphi^*\omega(X_1, \ldots, X_k))(p) &= \omega_{\varphi(p)}(d\varphi_p(X_1(p)), \ldots, d\varphi_p(X_k(p))) \\
&= \omega_{\varphi(p)}(d\varphi_{\varphi^{-1}(\varphi(p))}(X_1(\varphi^{-1}(\varphi(p))), \ldots, d\varphi_{\varphi^{-1}(\varphi(p))}(X_k(\varphi^{-1}(\varphi(p)))) \\
&= \omega_{\varphi(p)}(((\varphi^{-1})^* X_1)_{\varphi(p)}, \ldots, ((\varphi^{-1})^* X_k)_{\varphi(p)}) \\
&= ((\omega((\varphi^{-1})^* X_1, \ldots, (\varphi^{-1})^* X_k)) \circ \varphi)(p) \\
&= \varphi^*(\omega((\varphi^{-1})^* X_1, \ldots, (\varphi^{-1})^* X_k))(p),
\end{aligned}$$

since for any function, $g \in C^\infty(M)$, we have $\varphi^* g = g \circ \varphi$.

We know that

$$X_p[f] = \lim_{t \longrightarrow 0} \frac{(\Phi_t^* f)(p) - f(p)}{t}$$

and for any vector field, Y,

$$[X, Y]_p = (L_X Y)_p = \lim_{t \longrightarrow 0} \frac{\left(\Phi_t^* Y\right)_p - Y_p}{t}.$$

Since the one-parameter group associated with $-X$ is Φ_{-t} (this follows from $\Phi_{-t} \circ \Phi_t = \mathrm{id}$), we have

$$\lim_{t \longrightarrow 0} \frac{\left(\Phi_{-t}^* Y\right)_p - Y_p}{t} = -[X, Y]_p.$$

Now, using $\Phi_t^{-1} = \Phi_{-t}$ and $(*)$, we have

$$\begin{aligned}
(L_X\omega)(Y_1, \ldots, Y_k) &= \lim_{t \longrightarrow 0} \frac{(\Phi_t^*\omega)(Y_1, \ldots, Y_k) - \omega(Y_1, \ldots, Y_k)}{t} \\
&= \lim_{t \longrightarrow 0} \frac{\Phi_t^*(\omega(\Phi_{-t}^* Y_1, \ldots, \Phi_{-t}^* Y_k)) - \omega(Y_1, \ldots, Y_k)}{t} \\
&= \lim_{t \longrightarrow 0} \frac{\Phi_t^*(\omega(\Phi_{-t}^* Y_1, \ldots, \Phi_{-t}^* Y_k)) - \Phi_t^*(\omega(Y_1, \ldots, Y_k))}{t} \\
&\quad + \lim_{t \longrightarrow 0} \frac{\Phi_t^*(\omega(Y_1, \ldots, Y_k)) - \omega(Y_1, \ldots, Y_k)}{t}.
\end{aligned}$$

Call the first term A and the second term B. Then, as

$$X_p[f] = \lim_{t \longrightarrow 0} \frac{(\Phi_t^* f)(p) - f(p)}{t},$$

we have

$$B = X[\omega(Y_1, \ldots, Y_k)].$$

As to A, we have

$$\begin{aligned}
A &= \lim_{t\longrightarrow 0} \frac{\Phi_t^*(\omega(\Phi_{-t}^* Y_1, \ldots, \Phi_{-t}^* Y_k)) - \Phi_t^*(\omega(Y_1, \ldots, Y_k))}{t} \\
&= \lim_{t\longrightarrow 0} \Phi_t^* \left(\frac{\omega(\Phi_{-t}^* Y_1, \ldots, \Phi_{-t}^* Y_k) - \omega(Y_1, \ldots, Y_k)}{t} \right) \\
&= \lim_{t\longrightarrow 0} \Phi_t^* \left(\frac{\omega(\Phi_{-t}^* Y_1, \ldots, \Phi_{-t}^* Y_k) - \omega(Y_1, \Phi_{-t}^* Y_2, \ldots, \Phi_{-t}^* Y_k)}{t} \right) \\
&\quad + \lim_{t\longrightarrow 0} \Phi_t^* \left(\frac{\omega(Y_1, \Phi_{-t}^* Y_2, \ldots, \Phi_{-t}^* Y_k) - \omega(Y_1, Y_2, \Phi_{-t}^* Y_3, \ldots, \Phi_{-t}^* Y_k)}{t} \right) \\
&\quad + \cdots + \lim_{t\longrightarrow 0} \Phi_t^* \left(\frac{\omega(Y_1, \ldots, Y_{k-1}, \Phi_{-t}^* Y_k) - \omega(Y_1, \ldots, Y_k)}{t} \right) \\
&= \sum_{i=1}^{k} \omega(Y_1, \ldots, -[X, Y_i], \ldots, Y_k).
\end{aligned}$$

When we add up A and B, we get

$$\begin{aligned}
A + B &= X[\omega(Y_1, \ldots, Y_k)] - \sum_{i=1}^{k} \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_k) \\
&= (L_X\omega)(Y_1, \ldots, Y_k),
\end{aligned}$$

which finishes the proof. $\square$

Part (2) of Proposition 8.14 shows that the Lie derivative of a differential form can be defined in terms of the Lie derivatives of functions and vector fields:

$$\begin{aligned}
(L_X\omega)(Y_1, \ldots, Y_k) &= L_X(\omega(Y_1, \ldots, Y_k)) - \sum_{i=1}^{k} \omega(Y_1, \ldots, Y_{i-1}, L_X Y_i, Y_{i+1}, \ldots, Y_k) \\
&= X[\omega(Y_1, \ldots, Y_k)] - \sum_{i=1}^{k} \omega(Y_1, \ldots, Y_{i-1}, [X, Y_i], Y_{i+1}, \ldots, Y_k).
\end{aligned}$$

The following proposition is known as *Cartan's Formula*:

Proposition 8.15 *(Cartan's Formula) Let M be a smooth manifold. For every vector field, $X \in \mathfrak{X}(M)$, for every $\omega \in \mathcal{A}^k(M)$, we have*

$$L_X\omega = i(X)d\omega + d(i(X)\omega),$$

that is, $L_X = i(X) \circ d + d \circ i(X)$.

Proof. If $k = 0$, then $L_X f = X[f] = df(X)$ for a function, f, and on the other hand, $i(X)f = 0$ and $i(X)df = df(X)$, so the equation holds. If $k \geq 1$, then we have

$$\begin{aligned}
(i(X)d\omega)(X_1, \ldots, X_k) &= d\omega(X, X_1, \ldots, X_k) \\
&= X[\omega(X_1, \ldots, X_k)] + \sum_{i=1}^{k} (-1)^i X_i[\omega(X, X_1, \ldots, \widehat{X_i}, \ldots, X_k)] \\
&\quad + \sum_{j=1}^{k} (-1)^j \omega([X, X_j], X_1, \ldots, \widehat{X_j}, \ldots, X_k) \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X, X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(di(X)\omega)(X_1, \ldots, X_k) &= \sum_{i=1}^{k} (-1)^{i-1} X_i[\omega(X, X_1, \ldots, \widehat{X_i}, \ldots, X_k)] \\
&\quad + \sum_{i<j} (-1)^{i+j} \omega(X, [X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k).
\end{aligned}$$

Adding up these two equations, we get

$$\begin{aligned}
(i(X)d\omega + di(X))\omega(X_1, \ldots, X_k) &= X[\omega(X_1, \ldots, X_k)] \\
&\quad + \sum_{i=1}^{k} (-1)^i \omega([X, X_i], X_1, \ldots, \widehat{X_i}, \ldots, X_k) \\
&= X[\omega(X_1, \ldots, X_k)] - \sum_{i=1}^{k} \omega(X_1, \ldots, [X, X_i], \ldots, X_k) = (L_X\omega)(X_1, \ldots, X_k),
\end{aligned}$$

as claimed. $\square$

The following proposition states more useful identities, some of which can be proved using Cartan's formula:

Proposition 8.16 *Let M be a smooth manifold. For all vector fields, $X, Y \in \mathfrak{X}(M)$, for all $\omega \in \mathcal{A}^k(M)$, we have*

(1) $L_X i(Y) - i(Y)L_X = i([X, Y])$.

(2) $L_X L_Y \omega - L_Y L_X \omega = L_{[X,Y]}\omega$.

(3) $L_X i(X)\omega = i(X)L_X\omega$.

(4) $L_{fX}\omega = fL_X\omega + df \wedge i(X)\omega$, *for all* $f \in C^\infty(M)$.

(5) *For any diffeomorphism, $\varphi\colon M \to N$, for all $Z \in \mathfrak{X}(N)$ and all $\beta \in \mathcal{A}^k(N)$,*

$$\varphi^* L_Z \beta = L_{\varphi^* Z} \varphi^* \beta.$$

Finally, here is a proposition about the Lie derivative of tensor fields. Obviously, tensor product and contraction of tensor fields are defined pointwise on fibres, that is

$$\begin{aligned} (\alpha \otimes \beta)_p &= \alpha_p \otimes \beta_p \\ (c_{i,j}\alpha)_p &= c_{i,j}\alpha_p, \end{aligned}$$

for all $p \in M$, where $c_{i,j}$ is the contraction operator of Definition 22.5.

Proposition 8.17 *Let M be a smooth manifold. For every vector field, $X \in \mathfrak{X}(M)$, the Lie derivative, $L_X\colon \Gamma(M, T^{\bullet,\bullet}(M)) \to \Gamma(M, T^{\bullet,\bullet}(M))$, is the unique local linear operator satisfying the following properties:*

(1) *$L_X f = X[f] = df(X)$, for all $f \in C^\infty(M)$.*

(2) *$L_X Y = [X, Y]$, for all $Y \in \mathfrak{X}(M)$.*

(3) *$L_X(\alpha \otimes \beta) = (L_X \alpha) \otimes \beta + \alpha \otimes (L_X \beta)$, for all tensor fields, $\alpha \in \Gamma(M, T^{r_1,s_1}(M))$ and $\beta \in \Gamma(M, T^{r_2,s_2}(M))$, that is, L_X is a derivation.*

(4) *For all tensor fields $\alpha \in \Gamma(M, T^{r,s}(M))$, with $r, s > 0$, for every contraction operator, $c_{i,j}$,*

$$L_X(c_{i,j}(\alpha)) = c_{i,j}(L_X \alpha).$$

The proof of Proposition 8.17 can be found in Gallot, Hullin and Lafontaine [60] (Chapter 1). The following proposition is also useful:

Proposition 8.18 *For every $(0, q)$-tensor, $S \in \Gamma(M, (T^*)^{\otimes q}(M))$, we have*

$$(L_X S)(X_1, \ldots, X_q) = X[S(X_1, \ldots, X_q)] - \sum_{i=1}^{q} S(X_1, \ldots, [X, X_i], \ldots, X_q),$$

for all $X_1, \ldots, X_q, X \in \mathfrak{X}(M)$.

There are situations in differential geometry where it is convenient to deal with differential forms taking values in a vector space. This happens when we consider connections and the curvature form on vector bundles and principal bundles and when we study Lie groups, where differential forms valued in a Lie algebra occur naturally.

8.4 Vector-Valued Differential Forms

Let us go back for a moment to differential forms defined on some open subset of $\mathbb{R}^n$. In Section 8.1, a differential form is defined as a smooth map, $\omega\colon U \to \bigwedge^p(\mathbb{R}^n)^*$, and since we have a canonical isomorphism,

$$\mu\colon \bigwedge^p(\mathbb{R}^n)^* \cong \mathrm{Alt}^p(\mathbb{R}^n;\mathbb{R}),$$

such differential forms are real-valued. Now, let F be any normed vector space, possibly infinite dimensional. Then, $\mathrm{Alt}^p(\mathbb{R}^n;F)$ is also a normed vector space and by Proposition 22.33, we have a canonical isomorphism

$$\mu\colon \left(\bigwedge^p(\mathbb{R}^n)^*\right)\otimes F \longrightarrow \mathrm{Alt}^p(\mathbb{R}^n;F).$$

Then, it is natural to define differential forms with values in F as smooth maps, $\omega\colon U \to \mathrm{Alt}^p(\mathbb{R}^n;F)$. Actually, we can even replace $\mathbb{R}^n$ with any normed vector space, even infinite dimensional, as in Cartan [30], but we do not need such generality for our purposes.

Definition 8.11 Let F by any normed vector space. Given any open subset, U, of $\mathbb{R}^n$, a smooth *differential p-form on U with values in F*, for short, *p-form on U*, is any smooth function, $\omega\colon U \to \mathrm{Alt}^p(\mathbb{R}^n;F)$. The vector space of all p-forms on U is denoted $\mathcal{A}^p(U;F)$. The vector space, $\mathcal{A}^*(U;F) = \bigoplus_{p\geq 0}\mathcal{A}^p(U;F)$, is the set of *differential forms on U with values in F*.

Observe that $\mathcal{A}^0(U;F) = C^\infty(U,F)$, the vector space of smooth functions on U with values in F and $\mathcal{A}^1(U;F) = C^\infty(U,\mathrm{Hom}(\mathbb{R}^n,F))$, the set of smooth functions from U to the set of linear maps from $\mathbb{R}^n$ to F. Also, $\mathcal{A}^p(U;F) = (0)$ for $p > n$.

Of course, we would like to have a "good" notion of exterior differential and we would like as many properties of "ordinary" differential forms as possible to remain valid. As will see in our somewhat sketchy presentation, these goals can be achieved except for some properties of the exterior product.

Using the isomorphism

$$\mu\colon \left(\bigwedge^p(\mathbb{R}^n)^*\right)\otimes F \longrightarrow \mathrm{Alt}^p(\mathbb{R}^n;F)$$

and Proposition 22.34, we obtain a convenient expression for differential forms in $\mathcal{A}^*(U;F)$. If $(e_1,\ldots,e_n)$ is any basis of $\mathbb{R}^n$ and $(e_1^*,\ldots,e_n^*)$ is its dual basis, then every differential p-form, $\omega \in \mathcal{A}^p(U;F)$, can be written uniquely as

$$\omega(x) = \sum_I e_{i_1}^*\wedge\cdots\wedge e_{i_p}^*\otimes f_I(x) = \sum_I e_I^*\otimes f_I(x) \qquad x\in U,$$

where each $f_I\colon U \to F$ is a smooth function on U. By Proposition 22.35, the above property can be restated as the fact every differential p-form, $\omega \in \mathcal{A}^p(U; F)$, can be written uniquely as

$$\omega(x) = \sum_I e^*_{i_1} \wedge \cdots \wedge e^*_{i_p} \cdot f_I(x), \qquad x \in U.$$

where each $f_I\colon U \to F$ is a smooth function on U.

As in Section 22.15 (following H. Cartan [30]) in order to define a multiplication on differential forms we use a bilinear form, $\Phi\colon F \times G \to H$. Then, we can define a multiplication, $\wedge_\Phi$, directly on alternating multilinear maps as follows: For $f \in \mathrm{Alt}^m(\mathbb{R}^n; F)$ and $g \in \mathrm{Alt}^n(\mathbb{R}^n; G)$,

$$(f \wedge_\Phi g)(u_1, \ldots, u_{m+n}) = \sum_{\sigma \in \mathrm{shuffle}(m,n)} \mathrm{sgn}(\sigma)\, \Phi(f(u_{\sigma(1)}, \ldots, u_{\sigma(m)}), g(u_{\sigma(m+1)}, \ldots, u_{\sigma(m+n)})),$$

where shuffle(m, n) consists of all (m, n)-"shuffles", that is, permutations, σ, of $\{1, \ldots m+n\}$, such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$.

Then, we obtain a multiplication,

$$\wedge_\Phi\colon \mathcal{A}^p(U; F) \times \mathcal{A}^q(U; G) \to \mathcal{A}^{p+q}(U; H),$$

defined so that, for any differential forms, $\omega \in \mathcal{A}^p(U; F)$ and $\eta \in \mathcal{A}^q(U; G)$,

$$(\omega \wedge_\Phi \eta)_x = \omega_x \wedge_\Phi \eta_x, \qquad x \in U.$$

In general, not much can be said about $\wedge_\Phi$ unless Φ has some additional properties. In particular, $\wedge_\Phi$ is generally not associative. In particular, there is no analog of Proposition 8.1. For simplicity of notation, we write $\wedge$ for $\wedge_\Phi$. Using Φ, we can also define a multiplication,

$$\cdot\colon \mathcal{A}^p(U; F) \times \mathcal{A}^0(U; G) \to \mathcal{A}^p(U; H),$$

given by

$$(\omega \cdot f)_x(u_1, \ldots, u_p) = \Phi(\omega_x(u_1, \ldots, u_p), f(x)),$$

for all $x \in U$ and all $u_1, \ldots, u_p \in \mathbb{R}^n$. This multiplication will be used in the case where $F = \mathbb{R}$ and $G = H$, to obtain a normal form for differential forms.

Generalizing d is no problem. Observe that since a differential p-form is a smooth map, $\omega\colon U \to \mathrm{Alt}^p(\mathbb{R}^n; F)$, its derivative is a map,

$$\omega'\colon U \to \mathrm{Hom}(\mathbb{R}^n, \mathrm{Alt}^p(\mathbb{R}^n; F)),$$

such that ω'_x is a linear map from $\mathbb{R}^n$ to $\mathrm{Alt}^p(\mathbb{R}^n; F)$, for every $x \in U$. We can view ω'_x as a multilinear map, $\omega'_x\colon (\mathbb{R}^n)^{p+1} \to F$, which is alternating in its last p arguments. As in Section 8.1, the exterior derivative, $(d\omega)_x$, is obtained by making ω'_x into an alternating map in all of its $p+1$ arguments.

Definition 8.12 For every $p \geq 0$, the *exterior differential*, $d\colon \mathcal{A}^p(U;F) \to \mathcal{A}^{p+1}(U;F)$, is given by

$$(d\omega)_x(u_1, \ldots, u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} \omega'_x(u_i)(u_1, \ldots, \widehat{u_i}, \ldots, u_{p+1}),$$

for all $\omega \in \mathcal{A}^p(U;F)$ and all $u_1, \ldots, u_{p+1} \in \mathbb{R}^n$, where the hat over the argument u_i means that it should be omitted.

For any smooth function, $f \in \mathcal{A}^0(U;F) = C^\infty(U,F)$, we get

$$df_x(u) = f'_x(u).$$

Therefore, for smooth functions, *the exterior differential, df, coincides with the usual derivative, f'.* The important observation following Definition 8.3 also applies here. If $x_i\colon U \to \mathbb{R}$ is the restriction of pr_i to U, then x'_i *is the constant map* given by

$$x'_i(x) = pr_i, \qquad x \in U.$$

It follows that $dx_i = x'_i$ is the constant function with value $pr_i = e^*_i$. As a consequence, every p-form, ω, can be uniquely written as

$$\omega_x = \sum_I dx_{i_1} \wedge \cdots \wedge dx_{i_p} \otimes f_I(x)$$

where each $f_I\colon U \to F$ is a smooth function on U. Using the multiplication, $\cdot$, induced by the scalar multiplication in F ($\Phi(\lambda, f) = \lambda f$, with $\lambda \in \mathbb{R}$ and $f \in F$), we see that every p-form, ω, can be uniquely written as

$$\omega = \sum_I dx_{i_1} \wedge \cdots \wedge dx_{i_p} \cdot f_I.$$

As for real-valued functions, for any $f \in \mathcal{A}^0(U;F) = C^\infty(U,F)$, we have

$$df_x(u) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)(u)\, e^*_i,$$

and so,

$$df = \sum_{i=1}^n dx_i \cdot \frac{\partial f}{\partial x_i}.$$

In general, Proposition 8.3 fails unless F is finite-dimensional (see below). However for any arbitrary F, a weak form of Proposition 8.3 can be salvaged. Again, let $\Phi\colon F \times G \to H$ be a bilinear form, let $\cdot\colon \mathcal{A}^p(U;F) \times \mathcal{A}^0(U;G) \to \mathcal{A}^p(U;H)$ be as defined before Definition 8.12 and let $\wedge_\Phi$ be the wedge product associated with Φ. The following fact is proved in Cartan [30] (Section 2.4):

Proposition 8.19 *For all $\omega \in \mathcal{A}^p(U; F)$ and all $f \in \mathcal{A}^0(U; G)$, we have*

$$d(\omega \cdot f) = (d\omega) \cdot f + \omega \wedge_\Phi df.$$

Fortunately, $d \circ d$ still vanishes but this requires a completely different proof since we can't rely on Proposition 8.2 (see Cartan [30], Section 2.5). Similarly, Proposition 8.2 holds but a different proof is needed.

Proposition 8.20 *The composition $\mathcal{A}^p(U; F) \xrightarrow{d} \mathcal{A}^{p+1}(U; F) \xrightarrow{d} \mathcal{A}^{p+2}(U; F)$ is identically zero for every $p \geq 0$, that is,*

$$d \circ d = 0,$$

or using superscripts, $d^{p+1} \circ d^p = 0$.

To generalize Proposition 8.2, we use Proposition 8.19 with the product, $\cdot$, and the wedge product, $\wedge_\Phi$, induced by the bilinear form, Φ, given by scalar multiplication in F, that, is $\Phi(\lambda, f) = \lambda f$, for all $\lambda \in \mathbb{R}$ and all $f \in F$.

Proposition 8.21 *For every p form, $\omega \in \mathcal{A}^p(U; F)$, with $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_p} \cdot f$, we have*

$$d\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge_F df,$$

where $\wedge$ is the usual wedge product on real-valued forms and $\wedge_F$ is the wedge product associated with scalar multiplication in F.

More explicitly, for every $x \in U$, for all $u_1, \ldots, u_{p+1} \in \mathbb{R}^n$, we have

$$(d\omega_x)(u_1, \ldots, u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} (dx_{i_1} \wedge \cdots \wedge dx_{i_p})_x (u_1, \ldots, \widehat{u_i}, \ldots, u_{p+1}) df_x(u_i).$$

If we use the fact that

$$df = \sum_{i=1}^{n} dx_i \cdot \frac{\partial f}{\partial x_i},$$

we see easily that

$$d\omega = \sum_{j=1}^{n} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_j \cdot \frac{\partial f}{\partial x_j},$$

the direct generalization of the real-valued case, except that the "coefficients" are functions with values in F.

The pull-back of forms in $\mathcal{A}^*(V, F)$ is defined as before. Luckily, Proposition 8.6 holds (see Cartan [30], Section 2.8).

Proposition 8.22 *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be two open sets and let $\varphi\colon U \to V$ be a smooth map. Then*

(i) $\varphi^*(\omega \wedge \eta) = \varphi^*\omega \wedge \varphi^*\eta$, for all $\omega \in \mathcal{A}^p(V; F)$ and all $\eta \in \mathcal{A}^q(V; F)$.

(ii) $\varphi^*(f) = f \circ \varphi$, for all $f \in \mathcal{A}^0(V; F)$.

(iii) $d\varphi^*(\omega) = \varphi^*(d\omega)$, for all $\omega \in \mathcal{A}^p(V; F)$, that is, the following diagram commutes for all $p \geq 0$:

$$\begin{tikzcd}
\mathcal{A}^p(V;F) \arrow[r, "\varphi^*"] \arrow[d, "d"'] & \mathcal{A}^p(U;F) \arrow[d, "d"] \\
\mathcal{A}^{p+1}(V;F) \arrow[r, "\varphi^*"] & \mathcal{A}^{p+1}(U;F).
\end{tikzcd}$$

Let us now consider the special case where F has finite dimension m. Pick any basis, $(f_1, \ldots, f_m)$, of F. Then, as every differential p-form, $\omega \in \mathcal{A}^p(U; F)$, can be written uniquely as

$$\omega(x) = \sum_I e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \cdot f_I(x), \qquad x \in U.$$

where each $f_I\colon U \to F$ is a smooth function on U, by expressing the f_I over the basis, $(f_1, \ldots, f_m)$, we see that ω can be written uniquely as

$$\omega = \sum_{i=1}^m \omega_i \cdot f_i,$$

where $\omega_1, \ldots, \omega_m$ are smooth real-valued differential forms in $\mathcal{A}^p(U; \mathbb{R})$ and we view f_i as the constant map with value f_i from U to F. Then, as

$$\omega_x'(u) = \sum_{i=1}^m (\omega_i')_x(u) f_i,$$

for all $u \in \mathbb{R}^n$, we see that

$$d\omega = \sum_{i=1}^m d\omega_i \cdot f_i.$$

Actually, because $d\omega$ is defined independently of bases, the f_i do not need to be linearly independent; any choice of vectors and forms such that

$$\omega = \sum_{i=1}^k \omega_i \cdot f_i,$$

will do.

Given a bilinear map, $\Phi\colon F\times G \to H$, a simple calculation shows that for all $\omega \in \mathcal{A}^p(U;F)$ and all $\eta \in \mathcal{A}^p(U;G)$, we have

$$\omega \wedge_\Phi \eta = \sum_{i=1}^{m}\sum_{j=1}^{m'} \omega_i \wedge \eta_j \cdot \Phi(f_i, g_j),$$

with $\omega = \sum_{i=1}^m \omega_i \cdot f_i$ and $\eta = \sum_{j=1}^{m'} \eta_j \cdot g_j$, where $(f_1, \ldots, f_m)$ is a basis of F and $(g_1, \ldots, g_{m'})$ is a basis of G. From this and Proposition 8.3, it follows that Proposition 8.3 holds for finite-dimensional spaces.

Proposition 8.23 *If F, G, H are finite dimensional and $\Phi\colon F \times G \to H$ is a bilinear map, then For all $\omega \in \mathcal{A}^p(U;F)$ and all $\eta \in \mathcal{A}^q(U;G)$,*

$$d(\omega \wedge_\Phi \eta) = d\omega \wedge_\Phi \eta + (-1)^p \omega \wedge_\Phi d\eta.$$

On the negative side, in general, Proposition 8.1 still fails.

A special case of interest is the case where $F = G = H = \mathfrak{g}$ is a Lie algebra and $\Phi(a, b) = [a, b]$, is the Lie bracket of $\mathfrak{g}$. In this case, using a basis, $(f_1, \ldots, f_r)$, of $\mathfrak{g}$ if we write $\omega = \sum_i \alpha_i f_i$ and $\eta = \sum_j \beta_j f_j$, we have

$$[\omega, \eta] = \sum_{i,j} \alpha_i \wedge \beta_j [f_i, f_j],$$

where, for simplicity of notation, we dropped the subscript, Φ, on $[\omega, \eta]$ and the multiplication sign, $\cdot$. Let us figure out what $[\omega, \omega]$ is for a one-form, $\omega \in \mathcal{A}^1(U, \mathfrak{g})$. By definition,

$$[\omega, \omega] = \sum_{i,j} \omega_i \wedge \omega_j [f_i, f_j],$$

so

$$\begin{aligned}
[\omega, \omega](u, v) &= \sum_{i,j} (\omega_i \wedge \omega_j)(u, v)[f_i, f_j] \\
&= \sum_{i,j} (\omega_i(u)\omega_j(v) - \omega_i(v)\omega_j(u))[f_i, f_j] \\
&= \sum_{i,j} \omega_i(u)\omega_j(v)[f_i, f_j] - \sum_{i,j} \omega_i(v)\omega_j(u)[f_i, f_j] \\
&= [\sum_i \omega_i(u) f_i - \sum_j \omega_j(v) f_j] - [\sum_i \omega_i(v) f_i - \sum_j \omega_j(u) f_j] \\
&= [\omega(u), \omega(v)] - [\omega(v), \omega(u)] \\
&= 2[\omega(u), \omega(v)].
\end{aligned}$$

Therefore,

$$[\omega, \omega](u, v) = 2[\omega(u), \omega(v)].$$

Note that in general, $[\omega, \omega] \neq 0$, because ω is vector valued. Of course, for real-valued forms, $[\omega, \omega] = 0$. Using the Jacobi identity of the Lie algebra, we easily find that

$$[[\omega, \omega], \omega] = 0.$$

The generalization of vector-valued differential forms to manifolds is no problem, except that some results involving the wedge product fail for the same reason that they fail in the case of forms on open subsets of $\mathbb{R}^n$.

Given a smooth manifold, M, of dimension n and a vector space, F, the set, $\mathcal{A}^k(M; F)$, of *differential k-forms on M with values in F* is the set of maps, $p \mapsto \omega_p$, with $\omega_p \in \left(\bigwedge^k T_p^* M\right) \otimes F \cong \mathrm{Alt}^k(T_p M; F)$, which vary smoothly in $p \in M$. This means that the map

$$p \mapsto \omega_p(X_1(p), \ldots, X_k(p))$$

is smooth for all vector fields, $X_1, \ldots, X_k \in \mathfrak{X}(M)$. Using the operations on vector bundles described in Section 7.3, we can define $\mathcal{A}^k(M; F)$ as the set of smooth sections of the vector bundle, $\left(\bigwedge^k T^* M\right) \otimes \epsilon_F$, that is, as

$$\mathcal{A}^k(M; F) = \Gamma\left(\left(\bigwedge^k T^* M\right) \otimes \epsilon_F\right),$$

where ϵ_F is the trivial vector bundle, $\epsilon_F = M \times F$. In view of Proposition 7.12 and since $\Gamma(\epsilon_F) \cong C^\infty(M; F)$ and $\mathcal{A}^k(M) = \Gamma\left(\bigwedge^k T^* M\right)$, we have

$$\begin{aligned}
\mathcal{A}^k(M; F) &= \Gamma\left(\left(\bigwedge^k T^* M\right) \otimes \epsilon_F\right) \\
&\cong \Gamma\left(\bigwedge^k T^* M\right) \otimes_{C^\infty(M)} \Gamma(\epsilon_F) \\
&= \mathcal{A}^k(M) \otimes_{C^\infty(M)} C^\infty(M; F) \\
&\cong \bigwedge_{C^\infty(M)}^k (\Gamma(TM))^* \otimes_{C^\infty(M)} C^\infty(M; F) \\
&\cong \mathrm{Alt}^k_{C^\infty(M)}(\mathfrak{X}(M); C^\infty(M; F)).
\end{aligned}$$

with all of the spaces viewed as $C^\infty(M)$-modules. Therefore,

$$\mathcal{A}^k(M; F) \cong \mathcal{A}^k(M) \otimes_{C^\infty(M)} C^\infty(M; F) \cong \mathrm{Alt}^k_{C^\infty(M)}(\mathfrak{X}(M); C^\infty(M; F)),$$

which reduces to Proposition 8.12 when $F = \mathbb{R}$. The reader may want to carry out the verification that the theory generalizes to manifolds on her/his own. In Section 11.1, we will consider a generalization of the above situation where the trivial vector bundle, ϵ_F, is replaced by any vector bundle, $\xi = (E, \pi, B, V)$, and where $M = B$.

In the next section, we consider some properties of differential forms on Lie groups.

8.5 Differential Forms on Lie Groups and Maurer-Cartan Forms

Given a Lie group, G, we saw in Section 5.2 that the set of left-invariant vector fields on G is isomorphic to the Lie algebra, $\mathfrak{g} = T_1G$, of G (where 1 denotes the identity element of G). Recall that a vector field, X, on G is left-invariant iff

$$d(L_a)_b(X_b) = X_{L_ab} = X_{ab},$$

for all $a, b \in G$. In particular, for $b = 1$, we get

$$X_a = d(L_a)_1(X_1).$$

which shows that X is completely determined by its value at 1. The map, $X \mapsto X(1)$, is an isomorphism between left-invariant vector fields on G and $\mathfrak{g}$.

The above suggests looking at left-invariant differential forms on G. We will see that the set of left-invariant one-forms on G is isomorphic to $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, as a vector space.

Definition 8.13 Given a Lie group, G, a differential form, $\omega \in \mathcal{A}^k(G)$, is *left-invariant* iff

$$L_a^*\omega = \omega, \qquad \text{for all } a \in G,$$

where $L_a^*\omega$ is the pull-back of ω by L_a (left multiplication by a). The left-invariant one-forms, $\omega \in \mathcal{A}^1(G)$, are also called *Maurer-Cartan forms*.

For a one-form, $\omega \in \mathcal{A}^1(G)$, left-invariance means that

$$(L_a^*\omega)_g(u) = \omega_{L_ag}(d(L_a)_gu) = \omega_{ag}(d(L_a)_gu) = \omega_g(u),$$

for all $a, g \in G$ and all $u \in T_gG$. For $a = g^{-1}$, we get

$$\omega_g(u) = \omega_1(d(L_{g^{-1}})_gu) = \omega_1(d(L_g^{-1})_gu),$$

which shows that ω_g is completely determined by its value at $g = 1$.

We claim that the map, $\omega \mapsto \omega_1$, is an isomorphism between the set of left-invariant one-forms on G and $\mathfrak{g}^*$.

First, for any linear form, $\alpha \in \mathfrak{g}^*$, the one-form, α^L, given by

$$\alpha_g^L(u) = \alpha(d(L_g^{-1})_gu)$$

is left-invariant, because

$$\begin{aligned}
(L_h^*\alpha^L)_g(u) &= \alpha_{hg}^L(d(L_h)_g(u)) \\
&= \alpha(d(L_{hg}^{-1})_{hg}(d(L_h)_g(u))) \\
&= \alpha(d(L_{hg}^{-1} \circ L_h)_g(u)) \\
&= \alpha(d(L_g^{-1})_g(u)) = \alpha_g^L(u).
\end{aligned}$$

Second, we saw that for every one-form, $\omega \in \mathcal{A}^1(G)$,

$$\omega_g(u) = \omega_1(d(L_g^{-1})_g u),$$

so $\omega_1 \in \mathfrak{g}^*$ is the unique element such that $\omega = \omega_1^L$, which shows that the map $\alpha \mapsto \alpha^L$ is an isomorphism whose inverse is the map, $\omega \mapsto \omega_1$.

Now, since every left-invariant vector field is of the form $X = u^L$, for some unique, $u \in \mathfrak{g}$, where u^L is the vector field given by $u^L(a) = d(L_a)_1 u$, and since

$$\omega_{ag}(d(L_a)_g u) = \omega_g(u),$$

for $g = 1$, we get $\omega_a(d(L_a)_1 u) = \omega_1(u)$, that is

$$\omega(X)_a = \omega_1(u), \qquad a \in G,$$

which shows that $\omega(X)$ is a constant function on G. It follows that for every vector field, Y, (not necessarily left-invariant),

$$Y[\omega(X)] = 0.$$

Recall that as a special case of Proposition 8.13, we have

$$d\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y]).$$

Consequently, for all left-invariant vector fields, X, Y, on G, for every left-invariant one-form, ω, we have

$$d\omega(X, Y) = -\omega([X, Y]).$$

If we identify the set of left-invariant vector fields on G with $\mathfrak{g}$ and the set of left-invariant one-forms on G with $\mathfrak{g}^*$, we have

$$d\omega(X, Y) = -\omega([X, Y]), \qquad \omega \in \mathfrak{g}^*,\ X, Y \in \mathfrak{g}.$$

We summarize these facts in the following proposition:

Proposition 8.24 *Let G be any Lie group.*

(1) The set of left-invariant one-forms on G is isomorphic to $\mathfrak{g}^*$, the dual of the Lie algebra, $\mathfrak{g}$, of G, via the isomorphism, $\omega \mapsto \omega_1$.

(2) For every left-invariant one form, ω, and every left-invariant vector field, X, the value of the function $\omega(X)$ is constant and equal to $\omega_1(X_1)$.

(3) If we identify the set of left-invariant vector fields on G with $\mathfrak{g}$ and the set of left-invariant one-forms on G with $\mathfrak{g}^*$, then

$$d\omega(X, Y) = -\omega([X, Y]), \qquad \omega \in \mathfrak{g}^*,\ X, Y \in \mathfrak{g}.$$

Pick any basis, $X_1, \ldots, X_r$, of the Lie algebra, $\mathfrak{g}$, and let $\omega_1, \ldots, \omega_r$ be the dual basis of $\mathfrak{g}^*$. Then, there are some constants, c_{ij}^k, such that

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k.$$

The constants, c_{ij}^k are called the *structure constants* of the Lie algebra, $\mathfrak{g}$. Observe that $c_{ji}^k = -c_{ij}^k$.

As $\omega_i([X_p, X_q]) = c_{pq}^i$ and $d\omega_i(X, Y) = -\omega_i([X, Y])$, we have

$$\begin{aligned}
\sum_{j,k} c_{jk}^i \omega_j \wedge \omega_k(X_p, X_q) &= \sum_{j,k} c_{jk}^i (\omega_j(X_p)\omega_k(X_q) - \omega_j(X_q)\omega_k(X_p)) \\
&= \sum_{j,k} c_{jk}^i \omega_j(X_p)\omega_k(X_q) - \sum_{j,k} c_{jk}^i \omega_j(X_q)\omega_k(X_p) \\
&= \sum_{j,k} c_{jk}^i \omega_j(X_p)\omega_k(X_q) + \sum_{j,k} c_{kj}^i \omega_j(X_q)\omega_k(X_p) \\
&= c_{p,q}^i + c_{p,q}^i = 2c_{p,q}^i,
\end{aligned}$$

so we get the equations

$$d\omega_i = -\frac{1}{2} \sum_{j,k} c_{jk}^i \omega_j \wedge \omega_k,$$

known as the *Maurer-Cartan equations*.

These equations can be neatly described if we use differential forms valued in $\mathfrak{g}$. Let ω_{MC} be the one-form given by

$$(\omega_{\mathrm{MC}})_g(u) = d(L_g^{-1})_g u, \qquad g \in G,\ u \in T_g G.$$

The same computation that showed that α^L is left-invariant if $\alpha \in \mathfrak{g}$ shows that ω_{MC} is left-invariant and, obviously, $(\omega_{\mathrm{MC}})_1 = \mathrm{id}$.

Definition 8.14 Given any Lie group, G, the *Maurer-Cartan form on G* is the $\mathfrak{g}$-valued differential 1-form, $\omega_{\mathrm{MC}} \in \mathcal{A}^1(G, \mathfrak{g})$, given by

$$(\omega_{\mathrm{MC}})_g = d(L_g^{-1})_g, \qquad g \in G.$$

Recall that for every $g \in G$, conjugation by g is the map given by $a \mapsto gag^{-1}$, that is, $a \mapsto (L_g \circ R_{g^{-1}})a$, and the adjoint map, $\mathrm{Ad}(g) \colon \mathfrak{g} \to \mathfrak{g}$, associated with g is the derivative of $L_g \circ R_{g^{-1}}$ at 1, that is, we have

$$\mathrm{Ad}(g)(u) = d(L_g \circ R_{g^{-1}})_1(u), \qquad u \in \mathfrak{g}.$$

Furthermore, it is obvious that L_g and R_h commute.

Proposition 8.25 *Given any Lie group, G, for all $g \in G$, the Maurer-Cartan form, ω_{MC}, has the following properties:*

(1) $(\omega_{\mathrm{MC}})_1 = \mathrm{id}_{\mathfrak{g}}$.

(2) For all $g \in G$,
$$R_g^* \omega_{\mathrm{MC}} = \mathrm{Ad}(g^{-1}) \circ \omega_{\mathrm{MC}}.$$

(3) The 2-form, $d\omega \in \mathcal{A}^2(G, \mathfrak{g})$, satisfies the Maurer-Cartan equation,
$$d\omega_{\mathrm{MC}} = -\frac{1}{2}[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}].$$

Proof. Property (1) is obvious.

(2) For simplicity of notation, if we write $\omega = \omega_{\mathrm{MC}}$, then
$$\begin{aligned}
(R_g^*\omega)_h &= \omega_{hg} \circ d(R_g)_h \\
&= d(L_{hg}^{-1})_{hg} \circ d(R_g)_h \\
&= d(L_{hg}^{-1} \circ R_g)_h \\
&= d((L_h \circ L_g)^{-1} \circ R_g)_h \\
&= d(L_g^{-1} \circ L_h^{-1} \circ R_g)_h \\
&= d(L_g^{-1} \circ R_g \circ L_h^{-1})_h \\
&= d(L_{g^{-1}} \circ R_g)_1 \circ d(L_h^{-1})_h \\
&= \mathrm{Ad}(g^{-1}) \circ \omega_h,
\end{aligned}$$
as claimed.

(3) We can easily express ω_{MC} in terms of a basis of $\mathfrak{g}$. if $X_1, \ldots, X_r$ is a basis of $\mathfrak{g}$ and $\omega_1, \ldots, \omega_r$ is the dual basis, then $\omega_{\mathrm{MC}}(X_i) = X_i$, for $i = 1, \ldots, r$, so ω_{MC} is given by
$$\omega_{\mathrm{MC}} = \omega_1 X_1 + \cdots + \omega_r X_r,$$
under the usual identification of left-invariant vector fields (resp. left-invariant one forms) with elements of $\mathfrak{g}$ (resp. elements of $\mathfrak{g}^*$) and, for simplicity of notation, with the sign $\cdot$ omitted between ω_i and X_i. Using this expression for ω_{MC}, a simple computation shows that the Maurer-Cartan equation is equivalent to
$$d\omega_{\mathrm{MC}} = -\frac{1}{2}[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}],$$
as claimed. □

In the case of a matrix group, $G \subseteq \mathrm{GL}(n, \mathbb{R})$, it is easy to see that the Maurer-Cartan form is given explicitly by
$$\omega_{\mathrm{MC}} = g^{-1} dg, \qquad g \in G.$$

Thus, it is a kind of logarithmic derivative of the identity. For $n = 2$, if we let

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

we get

$$\omega_{\mathrm{MC}} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta d\alpha - \beta d\gamma & \delta d\beta - \beta d\delta \\ -\gamma d\alpha + \alpha d\gamma & -\gamma d\beta + \alpha d\delta \end{pmatrix}.$$

Remarks:

(1) The quantity, $d\omega_{\mathrm{MC}} + \frac{1}{2}[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}]$ is the *curvature* of the *connection* ω_{MC} on G. The Maurer-Cartan equation says that the curvature of the Maurer-Cartan connection is zero. We also say that ω_{MC} is a *flat* connection.

(2) As $d\omega_{\mathrm{MC}} = -\frac{1}{2}[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}]$, we get

$$d[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}] = 0,$$

which yields

$$[[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}], \omega_{\mathrm{MC}}] = 0.$$

It is easy to show that the above expresses the Jacobi identity in $\mathfrak{g}$.

(3) As in the case of real-valued one-forms, for every left-invariant one-form, $\omega \in \mathcal{A}^1(G, \mathfrak{g})$, we have

$$\omega_g(u) = \omega_1(d(L_g^{-1})_g u) = \omega_1((\omega_{\mathrm{MC}})_g u),$$

for all $g \in G$ and all $u \in T_g G$ and where $\omega_1 \colon \mathfrak{g} \to \mathfrak{g}$ is a linear map. Consequently, there is a bijection between the set of left-invariant one-forms in $\mathcal{A}^1(G, \mathfrak{g})$ and $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$.

(4) The Maurer-Cartan form can be used to define the *Darboux derivative* of a map, $f \colon M \to G$, where M is a manifold and G is a Lie group. The Darboux derivative of f is the $\mathfrak{g}$-valued one-form, $\omega_f \in \mathcal{A}^1(M, \mathfrak{g})$, on M given by

$$\omega_f = f^* \omega_{\mathrm{MC}}.$$

Then, it can be shown that when M is connected, if f_1 and f_2 have the same Darboux derivative, $\omega_{f_1} = \omega_{f_2}$, then $f_2 = L_g \circ f_1$, for some $g \in G$. Elie Cartan also characterized which $\mathfrak{g}$-valued one-forms on M are Darboux derivatives ($d\omega + \frac{1}{2}[\omega, \omega] = 0$ must hold). For more on Darboux derivatives, see Sharpe [139] (Chapter 3) and Malliavin [101] (Chapter III).

8.6 Volume Forms on Riemannian Manifolds and Lie Groups

Recall from Section 7.4 that a smooth manifold, M, is a *Riemannian manifold* iff the vector bundle, TM, has a Euclidean metric. This means that there is a family, $(\langle -, - \rangle_p)_{p\in M}$, of inner products on each tangent space, T_pM, such that $\langle -, - \rangle_p$ depends smoothly on p, which can be expessed by saying that that the maps

$$x \mapsto \langle d\varphi_x^{-1}(e_i), d\varphi_x^{-1}(e_j)\rangle_{\varphi^{-1}(x)}, \qquad x \in \varphi(U),\ 1 \leq i, j \leq n$$

are smooth, for every chart, (U, φ), of M, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. We let

$$g_{ij}(x) = \langle d\varphi_x^{-1}(e_i), d\varphi_x^{-1}(e_j)\rangle_{\varphi^{-1}(x)}$$

and we say that the $n \times n$ matrix, $(g_{ij}(x))$, is the *local expression of the Riemannian metric on M at x* in the coordinate patch, (U, φ).

For orientability of manifolds, volume forms and related notions, please refer back to Section 3.8. If a Riemannian manifold, M, is orientable, then there is a volume form on M with some special properties.

Proposition 8.26 *Let M be a Riemannian manifold with* $\dim(M) = n$*. If M is orientable, then there is a uniquely determined volume form,* Vol_M*, on M with the following properties:*

(1) For every $p \in M$, for every positively oriented orthonormal basis $(b_1, \ldots, b_n)$ of T_pM, we have

$$\mathrm{Vol}_M(b_1, \ldots, b_n) = 1.$$

(2) In every orientation preserving local chart, (U, φ), we have

$$((\varphi^{-1})^*\mathrm{Vol}_M)_q = \sqrt{\det(g_{ij}(q))}\, dx_1 \wedge \cdots \wedge dx_n, \qquad q \in \varphi(U).$$

Proof. (1) Say the orientation of M is given by $\omega \in \mathcal{A}^n(M)$. For any two positively oriented orthonormal bases, $(b_1, \ldots, b_n)$ and $(b_1', \ldots, b_n')$, in T_pM, by expressing the second basis over the first, there is an orthogonal matrix, $C = (c_{ij})$, so that

$$b_i' = \sum_{j=1}^n c_{ij} b_j.$$

We have

$$\omega_p(b_1', \ldots, b_n') = \det(C)\omega_p(b_1, \ldots, b_n),$$

and as these bases are positively oriented, we conclude that $\det(C) = 1$ (as C is orthogonal, $\det(C) = \pm 1$). As a consequence, we have a well-defined function, $\rho\colon M \to \mathbb{R}$, with $\rho(p) > 0$ for all $p \in M$, such that

$$\rho(p) = \omega_p(b_1, \ldots, b_n),$$

for every positively oriented orthonormal basis, $(b_1, \ldots, b_n)$, of T_pM. If we can show that ρ is smooth, then $\mathrm{Vol}_M = \rho^{-1}\omega$ is the required volume form.

Let (U, φ) be a positively oriented chart and consider the vector fields, X_j, on $\varphi(U)$ given by

$$X_j(q) = d\varphi_q^{-1}(e_j), \qquad q \in \varphi(U),\ 1 \leq j \leq n.$$

Then, $(X_1(q), \ldots, X_n(q))$ is a positively oriented basis of $T_{\varphi^{-1}(q)}$. If we apply Gram-Schmidt orthogonalization we get an upper triangular matrix, $A(q) = (a_{ij}(q))$, of smooth functions on $\varphi(U)$ with $a_{ii}(q) > 0$ such that

$$b_i(q) = \sum_{j=1}^{n} a_{ij}(q) X_j(q), \qquad 1 \leq i \leq n,$$

and $(b_1(q), \ldots, b_n(q))$ is a positively oriented orthonormal basis of $T_{\varphi^{-1}(q)}$. We have

$$\begin{aligned} \rho(\varphi^{-1}(q)) &= \omega_{\varphi^{-1}(q)}(b_1(q), \ldots, b_n(q)) \\ &= \det(A(q))\omega_{\varphi^{-1}(q)}(X_1(q), \ldots, X_n(q)) \\ &= \det(A(q))(\varphi^{-1})^*\omega_q(e_1, \ldots, e_n), \end{aligned}$$

which shows that ρ is smooth.

(2) If we repeat the end of the proof with $\omega = \mathrm{Vol}_M$, then $\rho \equiv 1$ on M and the above formula yield

$$((\varphi^{-1})^*\mathrm{Vol}_M)_q = (\det(A(q)))^{-1} dx_1 \wedge \cdots \wedge dx_n.$$

If we compute $\langle b_i(q), b_k(q)\rangle_{\varphi^{-1}(q)}$, we get

$$\delta_{ik} = \langle b_i(q), b_k(q)\rangle_{\varphi^{-1}(q)} = \sum_{j=1}^{n}\sum_{l=1}^{n} a_{ij}(q) g_{jl}(q) a_{kl}(q),$$

and so, $I = A(q)G(q)A(q)^\top$, where $G(q) = (g_{jl}(q))$. Thus, $(\det(A(q)))^2 \det(G(q)) = 1$ and since $\det(A(q)) = \prod_i a_{ii}(q) > 0$, we conclude that

$$(\det(A(q)))^{-1} = \sqrt{\det(g_{ij}(q))},$$

which proves the formula in (2). $\square$

We saw in Section 3.8 that a volume form, ω_0, on the sphere $S^n \subseteq \mathbb{R}^{n+1}$ is given by

$$(\omega_0)_p(u_1, \ldots u_n) = \det(p, u_1, \ldots u_n),$$

where $p \in S^n$ and $u_1, \ldots u_n \in T_pS^n$. To be more precise, we consider the n-form, $\omega_0 \in \mathcal{A}^n(\mathbb{R}^{n+1})$ given by the above formula. As

$$(\omega_0)_p(e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}) = (-1)^{i-1} p_i,$$

where $p = (p_1, \ldots, p_{n+1})$, we have

$$(\omega_0)_p = \sum_{i=1}^{n+1} (-1)^{i-1} p_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Let $i\colon S^n \to \mathbb{R}^{n+1}$ be the inclusion map. For every $p \in S^n$, and every basis, $(u_1, \ldots, u_n)$, of T_pS^n, the $(n+1)$-tuple $(p, u_1, \ldots, u_n)$ is a basis of $\mathbb{R}^{n+1}$ and so, $(\omega_0)_p \neq 0$. Hence, $\omega_0 \restriction S^n = i^*\omega_0$ is a volume form on S^n. If we give S^n the Riemannian structure induced by $\mathbb{R}^{n+1}$, then the discussion above shows that

$$\mathrm{Vol}_{S^n} = \omega_0 \restriction S^n.$$

Let $r\colon \mathbb{R}^{n+1} - \{0\} \to S^n$ be the map given by

$$r(x) = \frac{x}{\|x\|}$$

and set

$$\omega = r^*\mathrm{Vol}_{S^n},$$

a closed n-form on $\mathbb{R}^{n+1} - \{0\}$. Clearly,

$$\omega \restriction S^n = \mathrm{Vol}_{S^n}.$$

Furthermore

$$\begin{aligned} \omega_x(u_1, \ldots, u_n) &= (\omega_0)_{r(x)}(dr_x(u_1), \ldots, dr_x(u_n)) \\ &= \|x\|^{-1} \det(x, dr_x(u_1), \ldots, dr_x(u_n)). \end{aligned}$$

We leave it as an exercise to prove that ω is given by

$$\omega_x = \frac{1}{\|x\|^n} \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

We know that there is a map, $\pi\colon S^n \to \mathbb{RP}^n$, such that $\pi^{-1}([p])$ consist of two antipodal points, for every $[p] \in \mathbb{RP}^n$. It can be shown that there is a volume form on $\mathbb{RP}^n$ iff n is even, in which case,

$$\pi^*(\mathrm{Vol}_{\mathbb{RP}^n}) = \mathrm{Vol}_{S^n}.$$

Thus, $\mathbb{RP}^n$ is orientable iff n is even.

Let G be a Lie group of dimension n. For any basis, $(\omega_1, \ldots, \omega_n)$, of the Lie algebra, $\mathfrak{g}$, of G, we have the left-invariant one-forms defined by the ω_i, also denoted ω_i, and obviously, $(\omega_1, \ldots, \omega_n)$ is a frame for TG. Therefore, $\omega = \omega_1 \wedge \cdots \wedge \omega_n$ is an n-form on G that is never zero, that is, a volume form. Since pull-back commutes with $\wedge$, the n-form ω is left-invariant. We summarize this as

Proposition 8.27 *Every Lie group, G, possesses a left-invariant volume form. Therefore, every Lie group is orientable.*